

## 1

## Black holes and branes in supergravity

## 1.1 Introduction

The goal of the present chapter is to introduce black holes and branes in supergravity in the simplest possible manner. As a result, we make no attempt to be complete and, in fact, we will purposely leave out many points dear to the hearts of practicing string theorists and super-gravity experts. In particular, spinors and supersymmetry will make only a passing appearance in section 1.2.3, which can be skipped if the material seems too technical (though be sure to read the Executive Summary at the end of that section). Instead, we focus on bosonic spacetime solutions and the dynamics of the associated bosonic fields.

Even within this limited scope, our referencing of the original works will be rather spotty. The interested reader can consult [1] for a more encyclopedic review of branes and black holes in string theory (as of 1997) with numerous references to the original works. We also refer the reader to [2] for a more recent review of black holes in 4- and 5-dimensional supergravity and for complimentary material in 10- and 11-dimensions, to [3] for a partial guide to the literature as of 2004, and to various textbooks [4, 5, 6, 7, 8, 9] for further reading.

Our treatment will focus on supergravity theories in 10- and 11-dimensions which are in many ways simpler than their lower-dimensional counterparts and which allow us to make direct contact with string theory. We draw heavily on Polchinski's treatment [5], though the style is (hopefully) more adapted to the current audience. We begin in section 1.2 with a discussion of both the kinematics and dynamics of 11-dimensional supergravity and then introduce the associated branes in section 1.3. We follow tradition in referring to 11-dimensional supergravity as “M-theory<sup>1</sup>.”

<sup>1</sup> The origin of this term nevertheless seems to be lost in the mists of history.

The case of ten dimensional supergravity is also central to our mission. We choose to approach this subject via Kaluza-Klein reduction of eleven dimensional supergravity. Section 1.4 thus begins with some introductory remarks on Kaluza-Klein compactifications and then constructs a supergravity theory (the type IIA theory) in 9+1 dimensions. Section 1.5 introduces the branes of type IIA theory and then discusses both the very similar case of type IIB supergravity and the so-called T-duality symmetry that relates the two theories. The type I and heterotic theories are mentioned only briefly. We close with some brief remarks on D-brane perturbation theory and a few words about black hole entropy via D-branes in section 1.6. We hope this provides a useful basis for the later chapters in this volume.

## 1.2 Supergravity in eleven dimensions

Before diving into the details, a few words of orientation are in order. We will shortly see that supergravity in eleven (10+1) dimensions is really not much more complicated than the 3+1 Einstein-Maxwell theory of Einstein-Hilbert gravity coupled to Maxwell electrodynamics. The same is not as true of supergravity in lower dimensions. In ten (9+1) dimensions and below, many interesting supergravity theories contain a so-called dilaton field which couples non-minimally to the Maxwell-like gauge fields. As a result, the equivalence principle does not hold in such theories and different fields couple to distinct metrics that differ by a conformal factor. However, in eleven dimensions, properties of the supersymmetry algebra guarantee that any supergravity theory containing no fields with spin higher than two<sup>2</sup> has no dilaton. In fact, there is a unique supergravity theory in eleven dimensions and it contains only three fields: the metric, a U(1) (i.e., abelian, Maxwell-like) gauge field, and a spin 3/2 gravitino.

In contrast, in  $D > 11$  spacetime dimensions there are *no* supergravity theories without fields of spin  $s > 2$ . The basic reason for this property is that supersymmetry is associated with generators having spin 1/2. The action of such ‘supercharges’ on a given field thus returns a field of different spin. Working through the details one finds that, in any dimension  $D$ , theories with only spins  $s \leq 2$  can accommodate no more than 32 supercharges. In asymptotically flat settings, asymptotic Lorentz symmetry implies that each supercharge is associated with some component of a spinor of the associated Lorentz group. It turns out that 11-dimensional Majorana spinors

<sup>2</sup> Except for anti-symmetric tensor fields, which propagate on curved manifolds without the constraints associated with other higher spin fields. Because of these constraints, theories with spin higher than two are generally believed to be inconsistent unless they include an infinite number of such fields.

have precisely 32 components, so that precisely one  $D = 11$  spinor worth of supercharges is allowed. Any supergravity theory with 32 supercharges is called maximally supersymmetric.

Another reason to begin in 11 dimensions is that lower dimensional supergravity theories can generally be obtained through the Kaluza-Klein mechanism in which some subset of the dimensions are taken to be compact and small (and by also using certain so-called “dualities”). This mechanism will be discussed in sections 1.4.1 and 1.4.2 below.

### 1.2.1 On $n$ -form gauge fields

We first address just the bosonic part of eleven-dimensional supergravity, setting the fermionic fields to zero. The differences between this truncated theory and 3+1 Einstein-Maxwell theory amount to just differing numbers of dimensions. This happens in two ways: The first is the obvious fact that the theory lives in a 10+1 spacetime instead of a 3+1 spacetime. The second is that the gauge field is itself slightly ‘larger’ than that of Maxwell theory. Instead of having a *vector* (or, equivalently, a one-form) potential, the potential is a 3-form:  $A_3$ .

We will encounter a number of  $n$ -form potentials below. Although they may at first seem unfamiliar, they are in fact a very natural (and very slight) generalization of Maxwell fields. An  $n$ -form gauge potential  $A_n$  is associated with an  $(n + 1)$ -form field strength of the form  $F_{n+1} = dA_n$ , where  $d$  is the exterior derivative. As a result, the field strength satisfies a Bianchi identity  $dF_{n+1} = 0$ . As with the familiar Maxwell field, there is an associated set of gauge transformations

$$A_n \rightarrow A_n + d\Lambda_{n-1}, \quad (1.1)$$

where  $\Lambda_{n-1}$  is an arbitrary  $(n - 1)$  form. Such gauge transformations leave the field strength  $F_{n+1}$  invariant. For reference purposes, we record our conventions for differential forms (which agree with [5]):

$$A_n = \frac{1}{p!} A_{\alpha_1 \dots \alpha_n} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_n}, \quad (1.2)$$

so that we have

$$\int A_n = \int A_{0123 \dots (n-1)} d^n x. \quad (1.3)$$

In  $D$  spacetime dimensions (generally taken to be 11 in this section), the equation of motion for such a gauge field is typically of the form

$$d \star F_{D-(n+1)} = \star J_{D-n}, \quad (1.4)$$

where, in a slight abuse of notation,  $\star F_{D-(n+1)}$  denotes the  $D-(n+1)$  form that is the Hodge-dual of  $F_{(n+1)}$  defined by

$$\star F_{\alpha_1 \dots \alpha_{d-p}} = \frac{1}{p!} \epsilon_{\alpha_1 \dots \alpha_{d-p}}^{\beta_1 \dots \beta_p} F_{\beta_1 \dots \beta_p}, \quad (1.5)$$

and  $\epsilon$  is the Levi-Civita tensor. Similarly,  $\star J_{D-n}$  is the Hodge-dual of some  $n$ -form current  $J_n$ . These conventions are consistent with taking the natural coupling in an action between an  $n$ -form gauge field and its current to be of the form  $\int_{\mathcal{M}} A_n \wedge \star J_{D-n} \propto \int_{\mathcal{M}} \sqrt{-g} A_{\alpha_1 \dots \alpha_n} J^{\alpha_1 \dots \alpha_n}$ , where  $\mathcal{M}$  denotes the spacetime manifold.

As usual, gauge symmetry implies that the current is conserved. However, current conservation for a  $(D-n)$ -form current with  $D-n > 1$  is, in a certain sense, a much stronger statement than conservation of the current in 3+1 Maxwell theory. Note that the analogue of Gauss' law in the present context is to define the charge  $Q_B$  contained in a  $(D-n)$ -ball  $B$  by the integral  $Q_{D-n} = \int_{\partial B} \star F_{D-(n+1)}$  over the boundary  $\partial B$  of that ball. Now, suppose that the current  $J_{D-n}$  in fact vanishes in a neighborhood of the surface  $\partial B$ . Then by Stokes theorem and equation (1.4) we can deform the surface  $\partial B$  in any way we like and, as long as the surface does not encounter any current, the total charge  $Q_{D-n}$  does not change.

Now, in familiar 3+1 Maxwell theory, electric charge is measured by integrals over 2-surfaces. This is associated with the fact that an electrically charged particle sweeps out a worldline in spacetime. Note that any sphere which can be collapsed to a point without encountering the worldline of the particle must enclose zero net charge. The important fact is that, in four dimensions, there are two-spheres which 'link' with any curve and which cannot in fact be shrunk to a point without encountering the particle's worldline. In contrast, circles do not link with worldlines in 3+1 dimensions. For this reason, particles in 3+1 dimensions cannot be electrically charged under any gauge field whose field strength is, for example, a 3-form. This illustrates a general relation between a gauge field and the associated charges: unless the world-volume of an object can link with surfaces of dimension  $D-(n+1)$ , it cannot be electrically charged under an  $n$ -form gauge potential  $A_n$ .

While we are here, we may as well work out the relevant counting in general. Let us suppose that we have an  $n$ -form gauge field  $A_n$  in  $D$  spacetime dimensions. Then, we must integrate  $\star F_{D-(n+1)}$  over a  $D-(n+1)$  surface in order to calculate the charge. Now, in  $D$  dimensions, surfaces of dimensions  $k$  and  $m$  can link without intersecting if  $k+m+1=D$  (i.e., curves and curves in three dimensions, 2-surfaces and worldlines in four dimensions, etc.). Thus, non-zero electric charge of  $A_n$  is associated with  $n$  dimensional

worldvolumes. Such objects are generically known as ‘ $p$ -branes’ (as higher dimensional generalizations of the term membrane). Here,  $p$  is the number of spatial dimensions of the object; i.e., the electric charge of an  $n$ -form gauge potential is carried by  $(n - 1)$ -branes, whose world volume has  $n - 1$  spatial dimensions and time. This is how strings, membranes, and other branes will arise in our discussion of supergravity.

Note that, although  $p$ -branes are extended objects, the concept of a charge *density* of  $A_{p+1}$ -charge on a  $p$ -brane is not appropriate. Recall that the charge is measured by any  $D - (p + 2)$  surface surrounding the brane and that, by the above charge conservation argument, smooth deformations of the surface do not change the charge so measured. Thus, the equations of motion tell us that moving the  $D - (p + 2)$  surface *along* the brane cannot *ever* change the flux through the surface; see Fig. 11.1 below. Thus, non-uniform ‘pure’  $p$ -branes cannot exist! The proper concept here is to assign to such a  $p$ -brane only one number, the total charge. It simply happens that the particular type of charge being measured is somewhat less local than the familiar electric charge; it is fundamentally associated with  $p + 1$  dimensional hypersurfaces in the spacetime.<sup>3</sup>

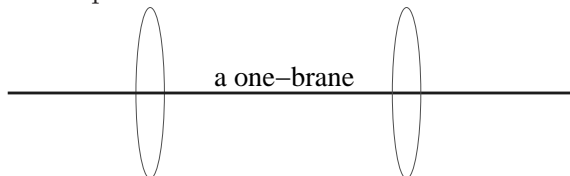


Figure 11.1 By charge conservation, both circles necessarily capture the same flux.

As a small complication, we will be interested not only in electric charges, but also in magnetic charges. Indeed, in supersymmetric string theory, both electric and magnetic charges appear to be on an equal footing. A useful point of contact for the present discussion is to realize that, in a certain sense, both electrically and magnetically charged ‘objects’ occur in pure Einstein-Maxwell theory without any matter fields. These are just the electrically and magnetically charged eternal black hole solutions. Although the Maxwell field satisfies both  $dF = 0$  and  $d \star F = 0$  at every non-singular point of such spacetimes, the black holes can still be said to ‘carry charge’ due to topological effects: the electric or magnetic flux starts in one asymptotic region, funnels through the Einstein-Rosen bridge at the ‘throat’ of the black hole, and out into the other asymptotic region<sup>4</sup>. Note that black holes

<sup>3</sup> Of course, it is possible for a  $p$ -brane to also carry  $A_n$ -form charges with  $n < p + 1$ . E.g., a string can carry point-charges in addition to some intrinsic string charge. In this case the proper concept is of a density of  $A_n$ -charge per unit  $(p - n - 1)$ -dimensional volume. Such charge densities need not be homogeneous.

<sup>4</sup> As a result, the electric charge of the black hole measured in one asymptotic region is the

(i.e., point-like or zero-brane objects) may carry both electric *and* magnetic charge for the Maxwell field in 3+1 dimensions.

The counting of dimensions for magnetic charges proceeds much like the counting for electric charges. To define what we mean by a magnetic charge, we recall that hodge duality  $F \rightarrow \star F$  in Maxwell theory interchanges electric and magnetic charge. Thus, since electric charge is associated with integrals of  $\star F_{D-(n+1)}$ , magnetic charge is defined by integrating the field strength  $F_{n+1}$  itself over an  $n+1$  surface. In  $D$  dimensions, an  $n+1$  surface can link with  $D-n-2$  worldvolumes, or  $(D-n-3)$ -branes. As a check, for 3+1 Maxwell theory, we have magnetic  $4-1-3=0$ -branes.

Let's take a look at the eleven-dimensional context. Without knowing anything more about supergravity than we already do, we can expect two types of 'objects' to be of particular interest from the point of view of the 3-form gauge field  $A_3$ . There may be 2+1 electrically charged objects (2-branes) and  $(D-n-3)+1 = (11-3-3)+1 = 5+1$  dimensional magnetically charged objects (5-branes). Since there are no explicit charges in the theory, these 'objects' (if they exist) must be black-hole like 'solitonic' solutions. We will see below that black two-brane and black five-brane solutions carrying the proper charges do indeed exist in eleven-dimensional supergravity. What is more, and what is different from lower dimensional supergravity, is that the horizons of these black branes remain smooth in the extreme limit of maximal electric or magnetic charge. The extremal versions of these brane-solutions are what are usually referred to as 'the M-theory two-brane' or 'M2-brane,' and 'the M-theory five-brane' or 'M5-brane.' We will discuss these in more detail in section 1.3.2.

### 1.2.2 Dynamics

I hope the discussion of section 1.2.1 has provided some orientation to supergravity in eleven dimensions. Now, however, it is time to fill in a few details. For example, it is appropriate to write down the full dynamics of the system. This is conveniently summarized by the action [10]

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \, e \left[ \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{2^3 \cdot 4!} (\bar{\psi}_\alpha \Gamma^{\alpha\beta\gamma\delta\sigma\lambda} \psi_\lambda + 12 \bar{\psi}^\beta \Gamma^{\gamma\delta} \psi^\sigma) (F + \hat{F})_{\beta\gamma\delta\sigma} \right]$$

opposite of the charge measured in the other asymptotic region. However, this need not trouble us so long as we understand that we must first orient ourselves by picking an asymptotic region in order to discuss the notion of charge.

$$- \bar{\psi}_\alpha \Gamma^{\alpha\beta\gamma} D_\beta \left( \frac{1}{2} (\omega + \hat{\omega}) \right) \psi_\gamma \Big] - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4, \quad (1.6)$$

where

$$|F_m|^2 = \frac{1}{m!} g^{\alpha_1\beta_1} \dots g^{\alpha_m\beta_m} F_{\alpha_1\dots\alpha_m} F_{\beta_1\dots\beta_m}. \quad (1.7)$$

Here  $R$  is the Ricci scalar of the metric  $g_{\alpha\beta}$ ,  $e_\alpha^a$  is the vielbein which squares to  $g_{\alpha\beta}$  (and  $e$  is its determinant),  $A_3$  is the three-form field discussed in the previous section, and  $\psi$  is the spin 3/2 gravitino. We use the notation

$$\begin{aligned} \hat{\omega}_{\alpha ab} &= \omega_{\alpha ab} + \frac{1}{8} \bar{\psi}^\beta \Gamma_{\beta\alpha ab} \psi^\gamma, \\ \hat{F}_{\alpha\beta\gamma\delta} &= F_{\alpha\beta\gamma\delta} - 3 \bar{\psi}_{[\alpha} \Gamma_{\beta\gamma} \psi_{\delta]}. \end{aligned} \quad (1.8)$$

In the above, Greek letters  $(\alpha, \beta, \dots)$  denote spacetime indices and Latin letters  $(a, b, \dots)$  denote internal indices. The square brackets [...] indicate a completely antisymmetric sum over permutations of the indices, divided by the number of terms. Our conventions for spinors and  $\Gamma$ -matrices are those of [4]. We will not state them explicitly here as spinors only make appearances in this section and section 1.2.3 and, in both cases, the details can be safely glossed over.

This looks a little complicated, but let's take a minute to sort through the various terms. We'll begin with the least familiar part: the gravitino. Since our attention here will be focused on classical solutions, we will be able to largely ignore the gravitino. The point here is that the gravitino is a fermion and, due to the Pauli exclusion principle, fermion fields do not have semi-classical states of the same sort that bosonic fields do. It is helpful here to think about the electron field as an example. There are, of course, states with a large number of electrons that are well described by a classical charged fluid. However, because of the exclusion principle, there are no semi-classical coherent states of the electron field itself; i.e., no states for which the dynamics is well described by a *classical spinor field*. In the same way, we might expect that there are states of the gravitino field that are well described by some sort of classical fluid, but we should only expect the classical action (1.6) to be a good description of the dynamics when the gravitino field vanishes. Thus, we will set  $\psi = 0$  throughout most of our discussion. This is self-consistent at the classical level as setting  $\psi = 0$  in the initial data is enough to guarantee  $\psi = 0$  for all time.

A study of (1.6) shows that the dynamics of the solutions for which  $\psi = 0$  can be obtained by simply setting  $\psi$  to zero in the action. This simplifies the situation sufficiently that it is worth rewriting the action as:

$$S_{\text{bosonic}} = \frac{1}{2\kappa_{11}^2} \int d^{11}x (-g)^{1/2} \left( R - \frac{1}{2} |F_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4. \quad (1.9)$$

This sort of presentation, giving only the bosonic terms, is quite common in the literature and is sufficient for most solutions of interest<sup>5</sup>. Now that the gravitino has been set to zero, we see that our action contains only three terms: the Einstein (scalar curvature) term  $R$ , the Maxwell-like term  $|F_4|^2$ , and the remaining so-called ‘Chern-Simons term.’

Note that, although the Chern-Simons term contains the gauge potential  $A_3$ , it is invariant under gauge transformations  $A_3 \rightarrow A_3 + d\Lambda_2$ , at least under ‘small’ gauge transformations for which  $\Lambda$  is a well-defined smooth 2-form and  $\Lambda_2 \rightarrow 0$  sufficiently rapidly at any boundaries. This term is very interesting, and turns out to significantly modify the picture outlined in section 1.2.1.

Though we will be able to ignore these modifications for our purposes below, it is worth saying just a few words about them here. To this end, suppose that we couple a 3-form source  $J_3$  to (1.9) via a term of the form  $\int A_3 \wedge \star J_3$ . The  $F_4$  equation of motion is then of the form

$$d(\star F_4 + (\text{const}) A_3 \wedge F_4) = \star J_3. \quad (1.10)$$

suggesting that the relevant conserved 2-brane charge  $Q_2$  is given by integrating  $\star F_4 + (\text{const}) A_3 \wedge F_4$  over some closed 7-surface. The Bianchi identity on  $F_4$  means that  $Q_2$  is invariant under ‘small’ gauge transformations though, like the action,  $Q_2$  does change under the action of ‘large’ gauge transformations where  $\Lambda_2$  is not single-valued. In fact, in units of the fundamental charge (discussed below), one can show that  $Q_2$  shifts by an integer multiple of an associated 5-brane charge  $Q_5$  defined by integrating  $F_4$  over a 4-surface. This feature allows M2-branes to end on M5-branes. See e.g. [12] for a review of this phenomenon and [13] for further comments on charge in the presence of Chern-Simons terms.

The reason we can ignore the Chern-Simons term below is that we will consider only relatively simple configurations of branes. Specifically we note that since  $dF = 0$ , the Chern-Simons contribution to (1.10) is proportional to  $F_4 \wedge F_4$ . Furthermore, the Chern-Simons term makes no contribution at all to the equation of motion for the metric. Thus, whenever there are 4 or more linearly independent vectors  $k$  at each point such that  $k^\alpha F_{\alpha\beta\gamma\delta} = 0$ , we

<sup>5</sup> Typically, an interesting bosonic solution in fact corresponds to a quantum state in some supersymmetry multiplet. The supersymmetry algebra can often be used to construct spacetime solutions corresponding to other states in the multiplet in which the fermions are excited. See for example [11].



have  $F_4 \wedge F_4 = 0$  and the Chern-Simons term does not affect the equations of motion. For the cases we consider below, this property is satisfied as all of the non-vanishing components of  $F$  will lie in a subspace of dimension seven or less.

### 1.2.3 Supersymmetry and BPS states

Some comments are now in order on the subject of supersymmetry, so that we may introduce (and then use!) the concept of Bogomuln'yi-Prasad-Sommerfeld (BPS) states. Again, I would like to begin with a few heuristics to provide a rough perspective for students of 3+1 general relativity. We will see below that BPS solutions are closely related to extremal solutions; in particular, to extremally charged solutions. As a result, most of our intuition from extreme Reissner-Nordström solutions carries over to the general BPS case.

The setting for any discussion of BPS solutions is the class of supergravity solutions which satisfy either asymptotically flat or asymptotically Kaluza-Klein boundary conditions. We thus require the topology of the asymptotic region to be of the form  $(\mathbb{R}^n - \Sigma) \times Y$  for some compact set  $\Sigma \subset \mathbb{R}^n$  and some homogeneous manifold  $Y$ , though we will not specify further details of the boundary conditions here.

In the setting of pure gravity, one would expect such spacetimes to exhibit asymptotic symmetries that correspond to the Poincaré group in the appropriate number of spacetime dimensions together with the symmetries of  $Y$ . Some particular solutions in this class will even have Killing vectors which make some subgroup of the Poincaré group into an *exact* symmetry of the spacetime; e.g. the rotation subgroup in spherically symmetric cases.

Now, supersymmetry is best thought of as an (anti-commuting) extension of the diffeomorphism group. Indeed, diffeomorphisms form a subgroup of the supersymmetry gauge transformations and, in the asymptotically flat setting just described, the asymptotic Poincaré transformations will be a subgroup of the asymptotic supersymmetry transformations. Solutions that are invariant under a subgroup of the supersymmetry transformations containing non-trivial anti-commuting elements are said to have a ‘Killing spinor’  $\eta$  and are known as BPS (Bogomuln'yi-Prasad-Sommerfeld) solutions.

It turns out that  $\bar{\eta}\Gamma^\mu\eta$  is then a Killing vector field of the solution. Furthermore, it is everywhere non-spacelike (i.e., timelike or null).<sup>6</sup> Since for

<sup>6</sup> When the spacetime is asymptotically flat in all directions (i.e., the fields decay in an asymptotic region diffeomorphic to  $\mathbb{R}^n$  minus some compact set), this Killing field is in fact

non-extreme black holes any stationary Killing field becomes spacelike behind the horizon, it follows that any BPS black hole must be extreme.

Having oriented ourselves with this intuitive introduction, it is now time to examine the details of the eleven-dimensional supersymmetry transformations and their algebra. The infinitesimal supersymmetry transformations are in one-to-one correspondence with Grassmann valued (Majorana<sup>7</sup>) spinor fields  $\eta(x)$ . The transformation associated with  $\eta$  is given by

$$\begin{aligned}\delta e_\alpha^a &= \frac{1}{2}\bar{\eta}\Gamma^a\psi_\alpha, \\ \delta A_{\alpha\beta\gamma} &= -\frac{3}{2}\bar{\eta}\Gamma_{[\alpha\beta}\psi_{\gamma]}, \\ \delta\psi_\alpha &= D_\alpha(\hat{\omega})\eta + \frac{\sqrt{2}}{(4!)^2}(\Gamma_\alpha^{abcd} - 8e_\alpha^a\Gamma^{bcd})\eta\hat{F}_{abcd} \equiv \hat{D}_\alpha\eta, \quad (1.11)\end{aligned}$$

where the last line defines the supercovariant derivative  $\hat{D}_\alpha$  acting on the spinor  $\eta$ .

The details of the supersymmetry transformations are not particularly important for our purposes. What *is* important is the general structure. Note that the variation of the vielbein  $e$  involves the gravitino  $\psi$ , but then the variation of the gravitino involves the connection  $\hat{\omega}$  which contains derivatives of the vielbein. Similarly taking two variations of the gauge field  $A_3$ , we find terms involving derivatives of the gauge field. As a result, the proper second variations give just diffeomorphisms of the spacetime.

Recalling that the variation of  $A_3$  contains  $\psi$ , we also note that the first variation of the gravitino field involves a derivative of the spinor  $\eta$ . Thus, the second variation of  $A_3$  is something that involves the derivative of  $\eta$ . With the proper choice of spinors  $\eta$ , one can construct a second supersymmetry variation that gives just the usual gauge transformation  $A_3 \rightarrow A_3 + d\Lambda_2$  on the gauge field, where this  $\Lambda_2$  is built from the relevant  $\eta$ 's. Thus, both diffeomorphisms and gauge transformations are in fact contained in the spacetime supersymmetry algebra. The supersymmetry algebra can be thought of as a sort of 'square root' of the diffeomorphism and gauge algebras. The fact that diffeomorphisms and gauge transformations are expressed as squares leads to extremely useful positivity properties.

While we will not need the details of the local supersymmetry algebra below, it is useful to display the algebra of the asymptotic supercharges.

timelike, except perhaps on a Killing horizon [14]. The structure of the argument is very much like the Witten proof of the positive energy theorem [15].

<sup>7</sup> i.e., satisfying the reality condition  $\eta^* = B\eta$  where  $B = \Gamma^3\Gamma^5\ldots\Gamma^9$  and  $*$  denotes complex conjugation.

Just as for diffeomorphisms and gauge transformations, the asymptotic supersymmetries lead, in the asymptotically flat context, to conserved ‘supercharges.’ In fact, for the eleven dimensional case, there are several relevant notions of the asymptotic algebra. This is because there are interesting  $p$ -branes with several values of  $p$ . There are thus several interesting classes of asymptotically Kaluza-Klein structures associated with different choices of the homogeneous manifold  $Y = \mathbb{R}^{11-p}$ .

However, all of these algebras are rather similar. If  $Q$  is the generator of supersymmetry transformations, so that the asymptotic versions of the transformations above are generated by taking (super) Poisson brackets with  $Q\bar{\eta}$ , then the algebra associated with the  $p$ -brane case has the general form (see e.g. [16])

$$\{Q_A^I, \bar{Q}^{JB}\}_+ = -2P_\mu \Gamma_A^{\mu B} \delta^{IJ} - 2iZ^{IJ} \delta_A^B, \quad (1.12)$$

where we have used  $A, B$  for the internal spinor indices. Here,  $P_\mu$  are the momenta per unit  $p$ -volume and  $Z^{IJ}$  is an antisymmetric real matrix associated with the asymptotic gauge transformations. In particular, the eigenvalues of  $Z$  are of the form  $\pm iq$  where  $q$  is the appropriately normalized charge carried by the  $p$ -brane. Our notation reflects the fact that it is natural to split the SUSY generator  $Q$ , which is an eleven dimensional Majorana fermion, into a set of  $(11-p)$  dimensional fermions  $Q^I$ . Thus, the indices  $A, B$  take values appropriate to spinors in  $(11-d)$  dimensions.

The most important property of this algebra is that it implies the so-called BPS bound on masses and charges. To get an idea of how this arises, recall that while  $\bar{Q}Q$  is a Lorentz invariant, it is  $Q^\dagger Q$  that is a positive definite operator. Thus, a positivity condition should follow by writing the algebra in terms of  $Q^\dagger$  and  $Q$ . For simplicity, let us also choose an asymptotic Lorentz frame such that energy-momentum of the spacetime is aligned with the time direction:  $P_\mu = T\delta_{\mu 0}$ , where  $T$  is the brane tension, or mass per unit  $p$ -volume<sup>8</sup>. The algebra then takes the form

$$\{Q_A^I, Q^{\dagger JB}\}_+ = 2T\delta^{IJ} \delta_A^B + 2iZ^{IJ} \Gamma_A^{0B}. \quad (1.13)$$

It is useful to adopt the notation of quantum mechanics, even though we are considering classical spacetimes. Thus, we describe a spacetime by a state  $|\psi\rangle$  and we let the generators  $Q$  act on that state as  $Q|\psi\rangle$ . Contracting

<sup>8</sup> Simple relativistic branes have a fixed value of  $T$ . As a result, stretching the brane over an additional  $p$ -volume  $V_p$  increases the energy by  $TV_p$  and requires a corresponding amount of work. Thus  $T$  is also the tension in the more familiar sense of the force-density (per unit  $(p-1)$ -volume) required to stretch the brane.

the above relation (1.13) with  $\eta_{JB}$  and  $\eta_I^{\dagger A}$  for a set of spinor fields  $\eta_I$ , taking the expectation value in any state, and using the positivity of the inner product and the fact that the eigenvalues of  $\Gamma_A^{0B}$  are  $\pm 1$  yields the relation

$$T \geq |q|. \quad (1.14)$$

See [17] for a full classical supergravity derivation in the context of magnetic charge in eleven dimensions and [14] for a complete derivation in classical  $N = 2$  supergravity in four dimensions. See also [18] for details of the argument above in the four dimensional context.

This is the BPS bound. A spacetime in which this bound is saturated is called a BPS spacetime and the corresponding quantum states are known as BPS states. Note that, from our above argument, a state is BPS only if it is annihilated by one of the supersymmetry generators; that is, if the spacetime is invariant under the transformation (1.11) for some spinor  $\eta$ . The converse is also true; any asymptotically flat spacetime which is invariant under some nontrivial supersymmetry transformation is BPS. Given a solution  $s$  and a spinor  $\eta$  for which the transformation (1.11) vanishes on  $s$ , one says that  $\eta$  is a *Killing spinor* of  $s$ . Since the gravitino  $\psi$  vanishes for a bosonic solution, in this context we see from (1.11) that  $\eta$  is a Killing spinor whenever it is supercovariantly constant; i.e when it satisfies  $\hat{D}_\alpha \eta = 0$ .

The bound (1.14) is reminiscent of the extremality bound for Reissner-Nordström black holes. It turns out that the relationship is a strong one. Given the similarity of eleven dimensional supergravity to Einstein-Maxwell theory, it will come as no surprise that there is a supergravity theory in 3+1 dimensions that contains Einstein-Maxwell theory, together with a few extra fields. When the extra fields vanish on an initial slice, they remain zero for all time. Thus, Einstein-Maxwell theory is a ‘consistent truncation’ of the supergravity. In this context, the BPS bound and the extremality bound for charge coincide when there is no angular momentum. Thus, any asymptotically flat solution of Einstein-Maxwell theory with extremal charge and vanishing angular momentum can be lifted to a BPS solution of the supergravity.

In general, any BPS black hole solution will be extremal, though the converse is not always true. An important example occurs in four dimensions where all BPS states must have zero angular momentum. Thus, the 3+1 extreme Kerr solution is not BPS.

Now, in familiar 3+1 Einstein-Maxwell gravity, we are used to thinking of extreme black holes as being some sort of marginal and perhaps unphysical case. Indeed, it is an important part of black hole thermodynamics that

one cannot by any finite (classical) process transform a non-extreme black hole into an extreme black hole. Moreover, a real astrophysical black hole will quickly lose its charge due to interactions with the interstellar medium. Even in a pure vacuum, quantum field theory effects in the real world cause black holes to lose their charge and to evolve toward neutral black holes. However, this last statement is a consequence of the large charge to mass ratio of the electron. Due to the BPS bound discussed above, objects like the familiar electron do not exist in a theory with sufficient supersymmetry. As a result, BPS black holes do not discharge. Instead, non-extreme black holes decay *toward* extremality through the emission of Hawking radiation and (assuming that black holes have only a finite number of internal states) any non-extreme black hole will decay to an extreme black hole in a very large but finite time. Thus, extreme black holes are of central importance in understanding supersymmetric theories as they represent stable ‘ground states’ for black holes.

Now that we have come to terms with supersymmetry, we will proceed to ignore fermions completely in the sections below.

**Executive Summary:** If you decided to skip over the material involving spinors, the key point is that theories with enough supersymmetry have a so-called BPS bound. This means that (in appropriate units) the mass per unit volume of a  $p$ -brane must be greater or equal to than the associated  $A_{p+1}$ -form charge. Solutions that saturate this bound are called BPS solutions and have special properties. In particular, any BPS black hole or brane is extremal.

### 1.3 M-branes: The BPS solutions

Although we wish to focus on the eleven dimensional case, supersymmetry and supergravity can also be considered in less than eleven dimensions. For example, in 3+1 dimensions any asymptotically flat solution of Einstein-Maxwell theory with extremal charge and zero angular momentum can be lifted to a BPS solution of 3+1 supergravity. But this is just the class of Majumdar-Papapetrou solutions [19], which consist of some number of extreme Reissner-Nordström black holes in static equilibrium. Since the Majumdar-Papapetrou solutions are a more familiar analogue of the eleven dimensional M-brane solutions which we wish to discuss, we present a brief review in section 1.3.1 as an introduction to the world of M-branes. This material overlaps with and extends the discussion of Chapter 1. We then examine the M-branes themselves in section 1.3.2.

### 1.3.1 The 3+1 Majumdar-Papapetrou solutions

Recall that the Reissner-Nordström solution with mass  $M$  and charge  $Q$  takes the form

$$ds^2 = -\left(1 - \frac{2GM}{R} + \frac{GQ^2}{R^2}\right)dt^2 + \frac{1}{1 - \frac{2GM}{R} + \frac{GQ^2}{R^2}}dR^2 + R^2d\Omega_2^2, \quad (1.15)$$

with  $R$  the usual Schwarzschild radial coordinate,  $t$  the Killing time, and  $d\Omega_2^2$  the metric on the unit two-sphere. Here,  $Q$  and  $M$  are the charge and mass of the black hole, with  $Q$  measured in units of  $\sqrt{(mass)(length)}$  as is natural in classical mechanics with  $c = 1$  but  $G \neq 1$ ; i.e., so that Coulomb's law is  $F = Q^2/r^2$ . The factors of Newton's constant  $G$  have been left explicit for consistency with the rest of this exposition. The extremal situation is  $GM^2 = Q^2$  and, in this case, the solution is controlled by a single length scale  $r_0 = GM = \sqrt{G}Q$ . The metric simplifies to take the form

$$ds^2 = -(1 - r_0/R)^2 dt^2 + (1 - r_0/R)^{-2} dR^2 + R^2 d\Omega_2^2. \quad (1.16)$$

Let us take this opportunity to recall that the horizon at  $r = r_0$  lies an infinite proper distance away from any  $r > r_0$  along any surface of constant  $t$ . Since the size of the sphere is approximately constant ( $r_0$ ) over the entire region near the horizon one says that an extreme black hole has “an infinite throat.” In fact, the region near the horizon is just the so-called Bertotti-Robinson universe, 2-dimensional anti-de Sitter space ( $\text{AdS}_2$ ) times  $S^2$ . A simple way to see this is to let  $z = r_0(1 - r_0/R)^{-1}$  and to expand the metric in powers of  $1/z$ . One finds

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} + r_0^2 d\Omega_2^2 + O(z^{-4}). \quad (1.17)$$

The leading terms shown give precisely  $\text{AdS}_2 \times S^2$  in so-called Poincaré coordinates.

We now return to the full metric (1.16) and change to so-called isotropic coordinates in which the spatial part of the metric is conformally flat. Let  $r = R - r_0$ , so that the horizon lies at  $r = 0$ . Introducing the Cartesian coordinates  $x^i$  as usual on  $\mathbf{R}^3$ , we have

$$ds^2 = -f^{-2}dt^2 + f^2 \sum_{i=1}^3 dx^i dx^i, \quad (1.18)$$

where  $f = 1 + r_0/r$ . Similarly, the electro-magnetic potential is given by

$A_t = f^{-1}$  with the spatial components of  $A$  vanishing. As the function  $f$  satisfies Poisson's equation with a delta function source,

$$\partial_x^2 f := \sum_{i=1}^3 \partial_i \partial_i f = -4\pi \delta^{(3)}(x), \quad (1.19)$$

the solution for the extreme black hole takes a form similar to that seen in electrodynamics (except that the Poisson equation is for the *inverse* of the electrostatic potential). Note that the relevant differential operator is the Laplacian on a *flat* three-space and not the one directly defined by the metric. Such differential operators will often appear below, and we will use the convention that  $\partial_x^2$  will always denote the flat-space Laplacian associated with the coordinates  $x$ . Similarly, we will write  $dx^2 := \sum_i dx^i dx^i$ .

The analogy with electrostatics is quite strong. The above metric (1.18) and the associated electric field define the class of Majumdar-Papapetrou solutions [19]. These are, in general, solutions of the Einstein-Maxwell system coupled to so-called extremal dust. This dust is characterized by the property that, when two grains of dust are at rest, their electrostatic repulsion is exactly sufficient to balance their gravitational attraction and they remain at rest. Modulo the conditions below, any choice of the function  $f$  in (1.18) yields a static solution of the field equations corresponding to some distribution of this dust. For an asymptotically flat solution, we should take  $f$  to be of the form  $1 + Q/r$  near infinity. The one restriction on  $f$  is that  $\rho = -\frac{1}{4\pi} \nabla^2 f$  must be everywhere positive. In particular, we will take it to be of the form  $\rho_0 + \sum_{k=1}^N r_k \delta(x - x_k)$  where  $\rho_0$  is continuous. The density (defined with respect to the Cartesian coordinate system  $x_i$ ) of extremal dust is given by  $\rho_0$  and each delta function will result in the presence of an extremal black hole.

Extremality is quite important for the simple form of this class of solutions. It is only in the extreme limit that the repulsion induced by the electric charge can ‘cancel’ the gravitational attraction so that the solution can remain static. If one adds any additional energy to the solution, the non-linearities of gravity become more directly manifest.

Note that the source in (1.19) lies at the origin of the  $x$ -coordinates; i.e., at the horizon of the black hole. However, since the horizon of the black hole is in fact not just a single point in space,  $x = 0$  is clearly a coordinate singularity. This means that although the support of the delta function lies at  $x = 0$ , this should not be interpreted as the location of the black hole charge. Rather, the role of this delta function is to enforce a boundary condition on

the electric flux emerging from the black hole so that the hole does indeed carry the proper charge.

Of course, in 3+1 dimensions, we can also have magnetically charged black holes. In fact, we can have dyons, carrying both electric and magnetic charge. The corresponding extremal solutions are given directly by electro-magnetic duality rotations of the above solution.

For future reference we note that there is a similar set of solutions in 4+1 dimensions, though black holes in five dimensions can carry only electric charge. They take the form

$$ds^2 = -f^{-2}dt^2 + f \sum_{i=1}^4 dx^i dx^i = -f^{-2}dt^2 + f dx^2, \quad (1.20)$$

where  $\partial_x^2 f = -2\Omega_3(\rho_0 + \sum_{k=1}^N r_k^2 \delta(x - x_k))$ ,  $\Omega_3$  is the volume of the unit three-sphere,  $\rho_0$  is the charge per unit  $d^4x$  cell, and  $r_k$  is a length scale parametrizing the charge of the  $k$ -th extremal black hole. The fact that  $r_k^2$ , as opposed to  $r_k$ , appears as the source reflects the fact that the fundamental solution of Poisson's equation in four dimensions is of the form  $r^{-2}$ .

As we have already commented, there is a coordinate singularity at the black hole horizon. Thus, the isotropic form of the metric does not allow us to see to what extent the black hole, or even the horizon, is non-singular. However, if the black hole is to have a smooth horizon, then a necessary condition is that the horizon have non-zero (and finite) area. That this is true of the above metrics is easy to read off from (1.18) and (1.20) by realizing that the divergence of  $f^2$  or  $f$  cancels the  $r^2$  factor that arises in writing  $dx^2 = dr^2 + r^2 d\Omega^2$  in spherical coordinates. While this is certainly not a sufficient condition for smoothness of the horizon, it will serve as a useful guide below.<sup>9</sup>

<sup>9</sup> Interestingly, while the 3+1 Majumdar-Papapetrou solutions (1.18) are smooth (and in fact analytic [20]), the 4+1 solutions (1.20) are not [17, 21]. In even higher dimensions the corresponding solutions have curvature singularities on the would-be horizon [22].



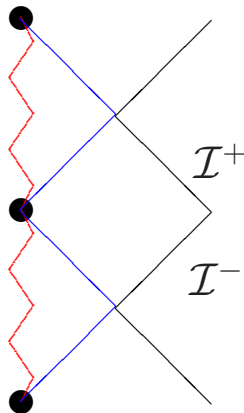


Figure 11.2 Conformal diagram for extreme Einstein-Maxwell black holes.

For completeness, we display the conformal diagrams for the above solutions in Fig. 11.2. The 3+1 and 4+1 cases are identical except for the dimension of the (suppressed) spheres of symmetry. Here  $\mathcal{I}^+$ ,  $\mathcal{I}^-$  denote the past and future null infinities of a particular asymptotic region and the wavy line on the right denotes the (timelike) singularity. The black circles mark the ‘internal infinities.’ These points lie at an infinite affine parameter along any geodesic (spacelike, timelike, or null) from the interior.

However, Fig. 11.2 does not tell the entire story. For charged non-extreme black holes it is known that, while the exterior solution is quite stable, the simple analytic textbook Reissner-Nordström solution is in fact unstable near the inner horizon. Perturbations transform the would-be inner horizon into a curvature singularity, see e.g. [23, 24, 25]. Now, recall that the inner horizon coincides with the event horizon in the extreme limit. Though there are some subtleties, this turns out to imply that extreme black holes (produced, say, by the emission of Hawking radiation from non-extreme black holes) have a singularity lurking just below their event horizon [26]. Specifically, in the limit where an observer enters the black hole long after the black hole was formed, no proper time passes for that observer between the horizon and the singularity. We say that such black holes have an ‘effective spacetime diagram’ with a singular horizon given by Fig. 11.3. We expect that the same is true of the higher dimensional extreme black holes discussed below, where by black *hole* we mean that the horizon is compactly generated. Black branes with noncompactly-generated horizons can be quite different due to the extra need to impose boundary conditions in the directions along the branes.

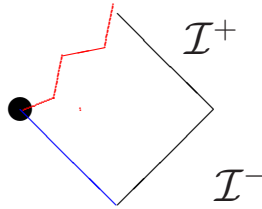


Figure 11.3 Effective conformal diagram for perturbed extreme Einstein-Maxwell black holes as seen by late-time observers.

### 1.3.2 Brane solutions in eleven dimensions

There are four (basic) solutions of eleven dimensional supergravity that are of particular importance in string/M-theory. They are known as the (eleven-dimensional) Aichelburg-Sexl metric [27], the M2-brane [28] (electrically charged under  $A_3$ ), the M5-brane (magnetically charged under  $A_3$ ), and the eleven dimensional version of the Kaluza-Klein monopole [29, 30]. Although it may not be obvious from the names, all four of the basic solutions are associated with branes in string/M-theory. Below, we discuss only the extreme versions of these basic solutions, which turn out to be BPS. In particular they each have 16 Killing spinors, preserving half of the supersymmetry. The non-extremal forms of the M-branes may be found in, e.g., [17].

It is sometimes said that an arbitrary BPS solution can be built from these basic solutions. The point here is that the above ‘basic’ solutions are in one-to-one correspondence with the types of charge present in eleven dimensional supergravity. Since the charges are additive, one is tempted to say that any solution with arbitrary amounts of the various charges can be built up by ‘combining’ these basic solutions. We will see below that certain simple solutions carrying multiple charge are in fact built from the basic solutions in a simple way. However, there is as yet no known method for writing down a general BPS solution at all, much less in terms of the basic solutions.

Let us begin with the BPS M2- and M5-branes as these are straightforward supergravity analogues of the extreme Reissner-Nordström black holes discussed above<sup>10</sup>. There is a corresponding notion of isotropic coordinates in which the multi-black hole solutions are given by solving a flat space

<sup>10</sup> Interestingly, the global structure of the *non-extreme* M2- and M5-brane solutions is much like that of the Schwarzschild black hole, as opposed to that of non-extreme Reissner-Nordström. In particular, there is no inner horizon and the singularity is spacelike as opposed to timelike.

Poisson equation with delta-function sources. The solutions of this Poisson equation are typically denoted  $H_2$  for the M2-brane and  $H_5$  for the M5-brane and are referred to as ‘harmonic’ functions. The details are different for the two branes, but both should seem quite familiar from our review of the Majumdar-Papapetrou solutions.

For the M2-brane, we introduce a set of three coordinates  $x_{\parallel}$  which should be thought of as labeling the directions along the brane, and a set of eight coordinates  $x_{\perp}$  which should be thought of as labeling the directions orthogonal to the brane. As one of the  $x_{\parallel}$  directions is the time direction, we define  $dx_{\parallel}^2 = -(dx_{\parallel}^0)^2 + (dx_{\parallel}^1)^2 + (dx_{\parallel}^2)^2$ . The solution takes the form:

$$\begin{aligned} A_3 &= H_2^{-1} dt \wedge dx_{\parallel,1} \wedge dx_{\parallel,2} \\ ds^2 &= H_2^{-2/3} dx_{\parallel}^2 + H_2^{1/3} dx_{\perp}^2, \end{aligned} \quad (1.21)$$

with  $\partial_{\perp}^2 H_2$  equal to a sum of delta-functions. Note that, near the delta function source,  $H_2$  will diverge like  $r^{-6}$ , where  $r$  is the  $x_{\perp}$  coordinate distance from the source. As a result,  $H_2^{1/3}$  diverges like  $r^{-2}$ , and the sphere at the horizon will have non-zero (finite) area. This suggests that the horizon of at least a single BPS M2-brane is smooth, and a careful investigation [17] does indeed show that this is the case. In fact, by a direct analogue of our discussion for 3+1 extreme Einstein-Maxwell black holes, one can show that the region near the horizon is just  $\text{AdS}_4 \times S^7$ .

This is rather interesting, as the extremal limits of black branes in lower dimensional supergravity theories tend, because of the dilaton, to have singular horizons. The global structure of the M2-brane is in fact much like that of the extreme Reissner-Nordström black holes discussed above. The conformal diagram is just that of Fig. 11.1, except that each point on the diagram now represents a surface with both the topology and metric of  $\mathbf{R}^2 \times S^7$  instead of just a sphere.

For the M5-brane, we introduce a set of six coordinates  $x_{\parallel}$  along the brane, and a set of five coordinates  $x_{\perp}$  orthogonal to the brane. Again, the  $x_{\parallel}$  directions include the time  $t$ . The solution takes the form:

$$\begin{aligned} dA &= F = -\frac{1}{4!} \partial_{x_{\perp}^i} H_5 \epsilon^i{}_{jklm} dx^j \wedge dx^k \wedge dx^l \wedge dx^m \\ ds^2 &= H_5^{-1/3} dx_{\parallel}^2 + H_5^{2/3} dx_{\perp}^2, \end{aligned} \quad (1.22)$$

with  $\partial_{\perp}^2 H_5$  equal to a sum of delta-functions. The different form of the gauge field as compared with (1.21) is associated with the fact that this solution carries a magnetic charge instead of an electric charge. Now the field  $H_5$

diverges at a delta-function source as  $r^{-3}$ , so that  $H_5^{2/3}$  diverges like  $r^{-2}$  and again the area of the spheres is finite at the horizon. Once again, a detailed study shows that the horizon is completely smooth and, as one might guess, the near-horizon geometry is  $\text{AdS}_7 \times S^4$ .

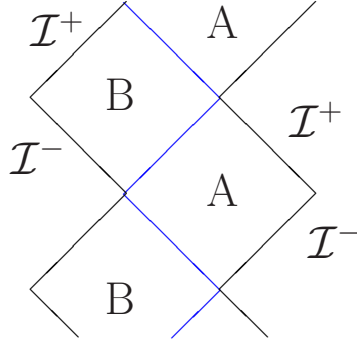


Figure 11.4 Conformal diagram for the extreme M5-brane.

The surprise [17] is that this solution turns out to have no singularities at all, even inside the horizon! Its conformal diagram is thus rather different from those we have encountered so far and is shown in Fig. 11.4. The regions marked A and B below (‘in front of’ and ‘behind’) the horizon are exactly the same. In familiar cases, the singularity theorems guarantee that something of this kind does not occur: compact trapped surfaces imply a singularity in their future [31]. However, the fact that we deal with a black brane, and not a black hole, means that the trapped surfaces are not in fact compact. The point here is that the horizon is extended in the  $x_{\parallel}$  directions. What happens when the solution is toroidally compactified by making identifications in the  $x_{\parallel}$  coordinates is an interesting story that will be discussed below.

The remaining two solutions have the interesting property of being BPS despite the fact that the gauge field  $A_3$  is identically zero. This is not really a contradiction to the condition of extremality when one notes (see section 1.4.2) that under Kaluza-Klein reduction a momentum can act like a charge. Another useful perspective results from recalling that two parallel beams of light (or two parallel gravitational waves) do not interact gravitationally. The same is true for any null particles. Thus, one may say that spatial components of the momentum (of like sign) provide a gravitational repulsion and that the case of null momentum is like the case of extremal charge, where this repulsion just exactly balances the gravitational attraction due to the energy of the particles.

The Aichelberg-Sexl metric carries just such a null momentum. This solution was originally constructed [27] (in 3+1 dimensions) by boosting a

Schwarzschild solution while rescaling its mass parameter  $M$  in order to keep the total energy  $E$  finite in some asymptotic frame. This explains the null momentum of the resulting solution. It too can be described in terms of a ‘harmonic function’  $H_{AS}$ . The Aichelberg-Sexl metric may be thought of as the gravitational field of a null *particle*, such as a graviton or a quantum of the  $A_3$  field in the short wavelength (WKB) approximation. We introduce a time coordinate  $t$ , a coordinate  $z$  in the direction of motion of the particle, and a set of nine additional coordinates  $x_\perp$ . In isotropic coordinates, the solution takes the form

$$ds^2 = -dt^2 + dx_\perp^2 + dz^2 + (H_{AS} - 1)(dt - dz)^2 \quad (1.23)$$

where  $H_{AS}(x_\perp)$  is a solution of  $\partial_\perp^2 H_{AS} = -7\Omega_9\rho$ , where  $\rho$  is again a source and  $\Omega_9$  is the volume of the unit 9-sphere. When  $\rho$  is a delta-function, this solution is in fact singular at the source.

Let us now turn to the Kaluza-Klein monopole. This solution was originally constructed [29, 30] by using the fact that the metric product of any two Ricci flat spaces is Ricci flat. Thus, one can make a static solution of 4+1 Einstein gravity out of any solution to four-dimensional, Euclidean gravity. Such a solution is Ricci flat, so the metric product with a line is also Ricci flat. The metric product of Euclidean Taub-NUT space [32] with a line gives the 4+1 Kaluza-Klein monopole. The eleven dimensional solution of interest here is simply the metric product of Euclidean Taub-NUT space with a 6+1 Minkowski space. Recall that the Taub-NUT solution is not asymptotically flat. Rather, it is asymptotically flat in three directions (to which we assign coordinates  $x_\perp$ ) and the fourth direction (which we will call  $\theta$ ) is an angular coordinate for which the associated  $S^1$  twists around the two-sphere to make a non-trivial asymptotic structure. Introducing coordinates  $x_\parallel$  on the 6+1 Minkowski space, the solution takes the form:

$$ds^2 = dx_\parallel^2 + H_{KK}dx_\perp^2 + H_{KK}^{-1}(d\theta + a_i dx_\perp^i)^2 \quad (1.24)$$

with, of course, vanishing 3-form potential  $A_3$ . Again,  $H_{KK}$  satisfies an equation of the form  $\partial_\perp^2 H_{KK} = -4\pi\rho$  and  $a_k$  is determined from  $H_{KK}$  via  $\partial_{x_\perp^i} H_{KK} = \epsilon_{ijk}\partial_{x_\perp^j} a_k$ . As usual, we find a coordinate singularity at the location of the delta-function sources.

But the story of this singularity is just that of Taub-NUT space. Suppose that  $\theta$  is periodic with period  $L$ . Then the spacetime is in fact smooth in the neighborhood of a ‘source’ of the form  $\rho = \frac{L}{4\pi}\delta^{(3)}(x_\perp)$ ; in this case, (1.24) actually represents a smooth geodesically complete solution to the *source-free* 10+1 Einstein equations. A source of this sort is referred to as a

monopole of unit charge. A multi-center solution<sup>11</sup> is smooth whenever each separate center has this charge. Now, if we take a limit of a multi-center solution in which several of the centers coalesce into a single center with charge greater than one, the resulting spacetime has a timelike singularity at the source. However, this singularity (with an integer  $n$  number of units of the above fundamental charge) has a particularly simple form. It is a quotient of flat space, and in this sense it is a higher dimensional version of a conical singularity.

A favorite topic to include in discussions of black holes is that of black hole entropy. It is therefore natural to ask about the entropy of the branes that we have discussed above. Similar thermodynamic arguments hold for black branes as for black holes, suggesting that one should associate an entropy of  $A/4G_{11}$  with such objects where  $A$  is the area (volume) of the horizon and  $1/16\pi G_{11} = 1/2\kappa_{11}^2$  is the coupling constant that stands in front of the supergravity action. However, the Kaluza-Klein monopole and Aichelberg-Sexl metric have no Killing horizons and so presumably carry no such entropy.

The M2- and M5-branes are a bit more subtle. On the one hand, their horizons are homogeneous surfaces that are non-compact. As such, one might be tempted to assign them infinite entropy. Some further insight into the issue is gained by using the fact that the solutions are invariant under translations in the spatial  $x_{\parallel}$  coordinates to make toroidal identifications and compactify the horizons. We can then calculate the horizon area and, because the norms of the Killing fields  $\partial_{x_{\parallel}}$  vanish on the horizon, the result is zero. Thus, at least when compactified in this way, the M2- and M5-branes have vanishing horizon volumes and again carry no entropy. This is another sense in which such solutions are ‘basic.’

One might guess that there is something singular about the zero-area horizons of the compactified M2- and M5-branes. However, since those solutions were constructed by making discrete identifications of spacetime with smooth horizons, the curvature and field strength cannot diverge at the zero-area horizon. It turns out that the situation is essentially the same as that which arises [33] when  $AdS_3$  is identified to make the  $M = 0$  BTZ black hole. The initially spacelike Killing fields  $\partial_{x_{\parallel}}$  become null on the horizon but also have fixed points. Thus, the horizon of the compactified solution has both closed null curves and a ‘Lorentzian conical singularity.’

<sup>11</sup> One with several delta-function sources.

### 1.3.3 Brane engineering

Before leaving eleven dimensions, a few words are in order on two of the basic techniques in ‘Brane Engineering,’ constructing new brane solutions from old. The particular techniques to be discussed are known as smearing and combining charges. Together, they will allow us to build BPS black branes with finite entropy.

Smearing is particularly straightforward. It is based on the observation that each type of ‘basic’ solution above is related to the solution of a linear differential equation. Using a delta-function source gives a solution which preserves some set of translation symmetries (in the  $x_{\parallel}$  directions) and breaks another set (in the  $x_{\perp}$  directions). However, a solution can be obtained that preserves more translational symmetries by using a more symmetric source, e.g., one supported on a line, plane, or a higher dimensional surface. Constructing such a solution can be thought of as ‘smearing out’ the charge of a less symmetric solution. Smearing out a given brane solution often results in a spacetime with a singular horizon. However, this need not be especially worrying if one regards the smeared solution as merely an effective description analogous to describing a collection of discrete atoms as a continuous fluid. One imagines an array of branes in which a large number of unsmeared basic branes are placed in the spacetime with a small spacing between the branes.

The next technique to discuss is that of combining the basic types of charge. As mentioned above, this is in general rather difficult. If, however, two solutions preserve some of the same supersymmetries *and* they have been engineered to have the same translation symmetries (for example, by smearing), then they tend to be rather easy to combine. Making a simple guess as to the way in which the relevant harmonic functions ( $H_2, H_5, H_{AS}, H_M$ ) should enter the metric and gauge fields tends to lead to a solution to the supergravity equations which preserves the common supersymmetries.

So far as I know, there are no general theorems available on this subject. We will thus content ourselves with a few simple examples which fall into the class discussed in [34, 35]; see also [2]. We have already discussed the solution (1.21) corresponding to a set of parallel M2-branes. This solution preserves half of the original 32 supersymmetries of 10+1 supergravity. The particular supersymmetries that are broken are related to the plane in space along which the M2-branes are oriented. Let us call the spatial coordinates along these branes  $x_{\parallel}^1$  and  $x_{\parallel}^2$ . We could also consider another set of M2-branes oriented along another plane associated with two other coordinates  $y_{\parallel}^1$  and  $y_{\parallel}^2$ , which are to be orthogonal to the  $x_{\parallel}$  coordinates. A set of solu-

tions containing both types of branes and preserving the 8 supersymmetries common to both sets of M2-branes separately is given by

$$\begin{aligned} A &= H_x^{-1} dt \wedge dx_{\parallel,1} \wedge dx_{\parallel,2} + H_y^{-1} dt \wedge dy_{\parallel,1} \wedge dy_{\parallel,2} \\ ds^2 &= -H_x^{-2/3} H_y^{-2/3} dt^2 + H_x^{-2/3} H_y^{1/3} dx_{\parallel}^2 + H_x^{1/3} H_y^{-2/3} dy_{\parallel}^2 \\ &\quad + H_x^{1/3} H_y^{1/3} dx_{\perp}^2, \end{aligned} \tag{1.25}$$

where  $H_x, H_y$  are functions only of the six spatial coordinates  $x_{\perp}$  that are transverse to both sets of branes. The functions  $H_x$  and  $H_y$  are, as usual, related to source distributions through  $\partial_{x_{\perp}}^2 H_x = -7\Omega_8 \rho_x$  and  $\partial_{x_{\perp}}^2 H_y = -7\Omega_8 \rho_y$  and the distributions  $\rho_x$  and  $\rho_y$  may be arbitrary functions of  $x_{\perp}$ .

Note that the form (1.25) is just like that of (1.21) except that we include two harmonic functions. A given term in the metric (1.25) is multiplied by a power of each harmonic function determined by whether the term refers to distances along or transverse to the corresponding brane. These powers are identical to the ones in (1.21).

In the solution (1.25), we have taken the two sets of branes to be completely orthogonal to each other. However, other choices of the relative angle still preserve the same amount of supersymmetry. If one thinks about the coordinates  $x_{\parallel}, y_{\parallel}$  as two sets of holomorphic coordinates on  $\mathbf{C}^2$ , then the requirement for a supersymmetric solution is that the  $x_{\parallel}$  and  $y_{\parallel}$  planes are related by a  $U(2)$  transformation [36] as opposed to a more general  $O(4)$  transformation. The metric in this case takes a similar form, with the part of the metric on the four-space spanned by  $x_{\parallel}, y_{\parallel}$  taking a certain Hermitian form.

However, combining the two sets of branes without first smearing them to generate four translation symmetries is more difficult. It turns out that, when one or both of the sets of branes is ‘localized’ (i.e., not completely spread out along the other set of branes) then the supergravity equations no longer cleanly divide into pieces describing each set of branes separately. The case where only one set of M2-branes is localized (and thus two translational symmetries remain) is still tractable, however. The solution still takes the same basic form (1.25) and construction of the solution still splits into two parts. One can first solve a standard flat space Poisson equation for the harmonic function  $H_x$  associated with the delocalized set of branes. One then has a linear differential equation to solve for the localized brane harmonic function  $H_y$ , where  $H_x$  appears in the particular differential operator to be inverted. Such solutions exhibit an interesting phenomenon: While localized solutions exist when the branes are separated (in the  $x_{\perp}$  directions), the



$y$ -branes effectively delocalize along the  $x$ -branes in the limit where this separation is removed<sup>12</sup>.

Let us now return to the smeared solution (1.25) and consider the case in which  $\rho_x = \rho_y = r_0^4 \delta^{(6)}(x_\perp)$ . We then find that  $H_x$  and  $H_y$  diverge at  $x_\perp = 0$  like  $|x_\perp|^{-4}$ . As a result, the 5-spheres at  $x_\perp = 0$  are infinite in volume and the solution is somewhat singular. However, adding a third M2-brane in another completely orthogonal  $(z_\parallel^1, z_\parallel^2)$  plane yields a non-singular solution. The metric and gauge field

$$\begin{aligned} A &= H_x^{-1} dt \wedge dx_{\parallel,1} \wedge dx_{\parallel,2} + H_y^{-1} dt \wedge dy_{\parallel,1} \wedge dy_{\parallel,2} + H_z^{-1} dt \wedge dz_{\parallel,1} \wedge dz_{\parallel,2} \\ ds^2 &= -H_x^{-2/3} H_y^{-2/3} H_z^{-2/3} dt^2 + H_x^{-2/3} H_y^{1/3} H_z^{1/3} dx_\parallel^2 + H_x^{1/3} H_y^{-2/3} H_z^{1/3} dy_\parallel^2 \\ &\quad + H_x^{1/3} H_y^{1/3} H_z^{-2/3} dz_\parallel^2 + H_x^{1/3} H_y^{1/3} H_z^{1/3} dx_\perp^2 \end{aligned} \quad (1.26)$$

for  $\partial_{x_\perp}^2 H_{x,y,z}(x_\perp) = -2\Omega_3 \rho_{x,y,z}(x_\perp)$  yield a BPS solution of the supergravity equations that preserves 1/8 of the supersymmetry (i.e., 4 supercharges) and has a smooth horizon. Moreover, this solution has the property that the translational Killing fields  $\partial x_\parallel^i, \partial y_\parallel^i, \partial z_\parallel^i$  have norms that do not vanish on the horizon.

In contrast, recall that while the solution (1.21) for a single M2-brane has a smooth horizon, the spatial translational Killing fields have vanishing norm there. As mentioned above, this means that compactifying a single M2-brane by, for example, taking the coordinate  $x_\parallel^1$  to live on a circle, yields a solution with a conical singularity at the horizon and vanishing entropy. On the other hand, because the norms of the spatial translations do not vanish for the solution (1.26), it compactifies nicely into a black object with finite horizon area<sup>13</sup>. This is the simplest BPS black brane solution with a finite entropy and, as a result, it is the simplest solution for which a microscopic accounting of the entropy has been given in string theory. A straightforward calculation shows the the horizon area is

$$A = \Omega_3 r_x r_y r_z L_{1x} L_{2x} L_{1y} L_{2y} L_{1z} L_{2z}, \quad (1.27)$$

where  $\Omega_3$  is the volume of the unit three-sphere and the  $L$ 's are the lengths of the various circles on which the solution has been compactified.

<sup>12</sup> This behavior is also typical of other BPS intersecting branes when the intersection (i.e., the set of directions common to both branes) is either 0+1 or 1+1 dimensional, though fully-localized solutions exist for higher-dimensional intersections. See [37, 38, 39] for details and [40, 41, 42] for examples of such fully localized solutions.

<sup>13</sup> As given by (1.26) this horizon is in fact smooth. We expect the compactified black hole to be subject to an instability similar to that of extreme Reissner-Nordstrom so that the perturbed conformal diagram is given by Fig. 11.3. Recall, however, that this instability affects only the black hole interior and so does not change the horizon area.

Now, charges are quantized in string/M- theory and it is useful to express the entropy in terms of the number of charge quanta  $Q_x, Q_y, Q_z$  carried by the various branes. The tension of a single M2-brane is  $(2\pi)^3 l_p^{-3}$ , where  $l_p$  is the eleven-dimensional Plank length, defined by  $16\pi G_{11} = 2\kappa_{11}^2 = (2\pi)^8 l_p^9$ . Note that  $r_x^2$  is a measure of the charge *density* of the  $x$ -type branes per unit cell of the  $y, z$  four-space. As such,  $r_x^2$  is proportional to  $Q_x/L_{1y}L_{2y}L_{1z}L_{2z}$ . Inspection of the area formula (1.27) thus shows that rewriting the area in terms of the integer charges will remove the factors of  $L$ . Putting in the proper normalization coefficients, the result turns out to be

$$A/4G_{11} = 2\pi\sqrt{Q_x Q_y Q_z}. \quad (1.28)$$

We will comment briefly on the corresponding microscopic counting of states in section 1.6.2.

#### 1.4 Kaluza-Klein and dimensional reduction

So far, we have dealt almost exclusively with eleven dimensional supergravity. However, it is the 10-dimensional supergravity theories that admit self-consistent perturbative quantizations in terms of strings<sup>14</sup>. This means that the powerful technology of perturbative quantum field theory can be brought to bear on questions concerning quantum dynamics. This perturbative technology can in particular be applied to certain branes in 9+1 supergravity. It is through this fusion of supergravity and perturbative field theory that string/M-theory has been revolutionized in recent years via studies of duality, black hole entropy, and more recently the Maldacena conjecture or AdS/CFT correspondence.

This article is not the place to enter into a detailed discussion of string perturbation theory, though we will comment briefly on the subject in section 1.6. The reader interested in learning that subject should consult the standard references (e.g. [4, 5, 6, 7, 8, 9]). Our purpose here is to provide a clear picture of the supergravity side of BPS brane physics, and in particular to discuss their relationship with the eleven dimensional theories. As the so-called type IIA 10-dimensional theory can be obtained by Kaluza-Klein reduction from M-theory, we warm-up by discussing Kaluza-Klein compactification in non-gravitational theories in section 1.4.1. We then address the reduction of M-theory to the IIA theory in detail in section 1.4.2, setting the stage for our discussion of 9+1 branes in section 1.5.

<sup>14</sup> It should be mentioned that string theory is not a quantization of pure 9+1 supergravity; string theory modifies the physics even at the classical level, though only by adding heavy fields with masses of order  $1/l_s$  where  $l_s$  is the string scale. See section 1.6.1 for (a few) more details.

### 1.4.1 Some remarks on Kaluza-Klein reduction

The idea of the Kaluza-Klein mechanism is that, at low energies, a *quantum* field theory on an  $n + d$ -dimensional spacetime in which  $d$  of the dimensions are compact behaves essentially like a quantum field theory on an  $n$ -dimensional spacetime. To see why, consider a free scalar field on  $M^n \times S^1$  where  $M^n$  is  $n$ -dimensional Minkowski space. Normal modes of the field are labeled by an  $n$ -vector momentum  $p$  and an integer  $k$  corresponding to momentum around the  $S^1$ . Suppose that the length of the  $S^1$  is  $L$ , so that the dispersion relation associated with one-particle excitations is  $E^2 = p^2 + (k/L)^2$ . If we now consider the theory at energy scales less than  $1/L$ , the only states with such a low energy have  $k = 0$ ; i.e., they are translationally invariant around the  $S^1$ .

In this way, our scalar field reduces at low energies to a quantum field on  $n$ -dimensional Minkowski space. Note that this is an intrinsically quantum mechanical effect, associated with both the quantization of energy and with the discrete spectrum of the Laplacian on a circle. Since the Laplacian has a discrete spectrum on any compact space, the same basic mechanism operates with any choice of compact manifold. The simplest cases to analyze are those in which the spacetime is a direct product of a non-compact spacetime  $M$  with a compact manifold  $K$ , and in which  $K$  is a homogeneous space. In that case, the lower dimensional (reduced) theory is typically obtained from the higher dimensional one simply by taking the fields to be invariant under the symmetry group that acts transitively on the compact manifold. On a general manifold of the form  $M \times K$ , the reduced theory is given by considering the zero-modes of the Laplacian (or other appropriate differential operator) on  $K$ . Similar, but less clean, mechanisms may apply even when the spacetime is not a direct product of a compact and a non-compact spacetime.

The effect of compactification on interacting fields is similar. At the perturbative level the story is exactly the same, and non-perturbative effects seldom change the picture significantly.

### 1.4.2 Kaluza-Klein in (super)gravity

We now turn to Kaluza-Klein reduction in a theory with gravity. Since the spacetime metric is dynamical, this case is perhaps not as clean cut as the scalar field example just discussed. However, at the perturbative level, one may treat gravity just as any other field. Our general experience with quantum mechanics and the uncertainty principle also makes it reasonable

on more general grounds to expect that excitations associated with the small compact space will be expensive in terms of energy. Thus, at least at first glance, we expect that gravity on a manifold of the form  $M \times K$  reduces at low energies to a theory on the non-compact manifold  $M$ .

We are most interested in Kaluza-Klein reduction of eleven-dimensional supergravity on  $\mathcal{M} = M \times S^1$  where  $M$  is a 9+1 dimensional asymptotically flat spacetime. We expect the reduced theory to be obtained by considering the class of eleven-dimensional field configurations that are translationally invariant around the  $S^1$ . Let us therefore assume that our eleven dimensional spacetime  $\mathcal{M}$  has a spacelike Killing vector field  $\lambda^\mu$  whose orbits have the topology  $S^1$ . It is convenient to normalize this Killing field to have norm +1 at infinity and to denote the length of the Killing orbits there by  $L$ . The Killing field is not necessarily hypersurface orthogonal.

Since the translation group generated by the Killing field acts nicely (technically, ‘properly discontinuously’ [31]) on our eleven dimensional spacetime  $\mathcal{M}$ , we may consider the quotient of the smooth topological space  $\mathcal{M}$  by the action of this group. The result is a new topological space  $M$ , which is a ten dimensional smooth manifold. This is the manifold on which our  $10 = 9 + 1$  dimensional reduced theory will live.

By using the metric, we define a set of projection operations on the various 10+1 fields with each projection providing a different field in the 9+1 dimensional spacetime. Recall that a field is an object which transforms in a certain way under local Lorentz transformations (i.e., diffeomorphisms) of the manifold. The diffeomorphisms of the 9+1 manifold will be that subgroup of the eleven-dimensional diffeomorphisms that leaves the killing field  $\lambda^\mu$  invariant. Thus, the transformations that become the diffeomorphisms of the 9 + 1 manifold are a proper subgroup of the 10+1 diffeomorphisms and a single 10+1 field can contain several 9+1 fields.

To see how the 9+1 fields are constructed, consider any coordinate patch  $U$  (with coordinates  $x^a$ ) on the 9+1 manifold  $M$ . If  $V \subset \mathcal{M}$  is the preimage of  $U$  under the above quotient construction, then each  $x^a$  defines a function on  $V$ . Since no linear combination of the gradients of the  $x^a$  functions can be proportional to the Killing field  $\lambda^\mu$ , we can complete this set of functions to a coordinate patch on  $V$  by adding a (periodic) coordinate  $\theta$  which is proportional to the Killing parameter along any orbit of  $\lambda^\mu$ ; i.e., satisfying  $\theta_{,\mu}\lambda^\mu = \lambda^\mu\lambda_\mu$ .

This coordinate system gives an explicit realization of the natural decomposition of the 10+1 fields into a set of 9+1 fields. The set of gradients  $x^a_{,\mu}$  of the 9+1 coordinates define a projection operation on any contravariant (upper) index, as does the gradient  $\theta_{,\mu}$  of the coordinate  $\theta$ . Thus, from the 10+1

contravariant metric  $g^{\mu\nu}$ , we can define the 9+1 metric  $g^{ab} = x^a_{,\mu} x^b_{,\nu} g^{\mu\nu}$ , a 9+1 abelian vector field  $A_1^a = -x^a_{,\mu} \theta_{,\nu} g^{\mu\nu}$ , and a 9+1 scalar field  $\phi$  through  $Le^{4\phi/3} = \lambda^\mu g_{\mu\nu} \lambda^\nu$ . The particular coefficient of  $\phi$  is chosen so that it is canonically normalized<sup>15</sup>. This  $\phi$  is the famous dilaton of string theory, and it is this field which is responsible for many of the differences between supergravity in less than eleven dimensions and familiar Einstein-Maxwell theory. It is clear that all of these fields transform in an appropriate way under 9+1 diffeomorphisms.

We now make several important observations. The first is that nondegeneracy of the 10+1 metric implies non-degeneracy of the 9+1 metric. Thus,  $g^{ab}$  has an inverse which gives the covariant metric  $g_{ab}$ .

The second is that the scalar has been defined by the norm of the Killing field and not the norm of  $\theta_{,\mu}$  as one might expect. The point is that these two objects are related. To see this, let us first note that the coordinates  $x^a$  are constant along the orbits of the Killing field. Thus, the Lie derivative of  $x^a$  along  $\lambda^\mu$  vanishes, and we have  $x^a_{,\nu} g^{\nu\mu} \lambda_\mu = 0$ . This means that the gradients  $x^a_{,\mu}$  span the space orthogonal to  $\lambda_\mu$  at each point. But, by definition,  $\lambda^\mu \theta_{,\mu} = \lambda^\mu \lambda_\mu$ . Thus, we find that  $\theta_{,\mu} - \lambda_\mu$  is of the form  $c_a x^a_{,\mu}$  where  $c_a$  is some function on the 9+1 spacetime. This fact, together with the definition of  $A_1$ , can be used to derive the relation:

$$c_a = -g_{ab} A_1^b. \quad (1.29)$$

Thus, we have

$$\theta_{,\mu} \theta^{,\mu} = \lambda^\mu \lambda_\mu + A_{1a} A_1^a. \quad (1.30)$$

We see that the definition of  $\phi$  differs from the seemingly more natural one only by a function of the vector field  $A_1$ . Choosing to write  $\phi$  directly in terms of the Killing field  $\lambda^\mu$  removes a mixing between the vector field and scalar that would otherwise obscure the physics. Note that we have related the scalar field  $\phi$  to the logarithm of the norm of the Killing field, and that this norm is positive by assumption.

Finally, let us consider the vector field  $A_1$ . Although we have  $x^a_{,\nu} g^{\nu\mu} \lambda_\mu = 0$ , the vector field  $A_1$  need not vanish. Note that there is a freedom to redefine the zero of  $\theta$  at each value of the  $x^a$ . This amounts to the transformation  $\theta \rightarrow \theta - \Lambda(x)$ . Under this operation, we see that the 9 + 1 metric  $g^{ab}$  is not affected, and neither is the scalar (since it depends only on the norm of the Killing field) while the vector field transforms as  $A_1^a \rightarrow A_1^a + \Lambda_{,b} g^{ab}$ ; i.e.,  $A_{1a} \rightarrow A_{1a} + \Lambda_{,a}$ . Thus, we see that  $A_1$  is in fact an abelian gauge field. The associated field strength  $F_2 = dA_1$  (with  $A_1$  considered as a 1-form) is just

<sup>15</sup> When  $2\kappa_{10}^2$  is set to one, see below.

the ‘twist’ of the Killing field  $\lambda^\mu$ , which measures the failure of  $\lambda^\mu$  to be hypersurface orthogonal.

It is interesting to ask about the charge to which this gauge field couples, as the field itself arose directly from the reduction of the gravitational field in eleven dimensions. To this end, consider a “gauge” transformation of the above form with constant  $\Lambda$ . In familiar Maxwell theory, this global  $U(1)$  rotation is generated by the total electric charge operator. But in terms of  $\mathcal{M}$  it is a shift of  $\theta$ , which is just a translation along  $\lambda^\mu$ . We therefore identify the total  $A_1$ -charge on  $M$  with the corresponding momentum on  $\mathcal{M}$ . One may check that any timelike total energy-momentum for  $\mathcal{M}$  becomes a charge and a ten-dimensional energy-momentum vector on  $M$  satisfying a BPS bound.

In performing calculations, it is often useful to express the above decomposition in terms of the eleven dimensional *covariant* metric  $ds_{11}^2$ . The reader may check that we have

$$ds_{11}^2 = g_{ab}dx^a dx^b + e^{4\phi/3}[d\theta + A_{1a}dx^a]^2. \quad (1.31)$$

One might think that it is natural to decompose the antisymmetric 3-form  $A_3$  into a 9+1 3-form  $\hat{A}_3^{abc} = A_3^{\mu\nu\rho}x_{,\mu}^a x_{,\nu}^b x_{,\rho}^c$  and a 2-form  $A_2^{ab} = A_3^{\mu\nu\rho}x_{,\mu}^a x_{,\nu}^b \lambda_\rho$  in order that both be invariant under the gauge transformation  $A_1 \rightarrow A_1 + d\Lambda_0$ . However, it turns out that the 3-form  $\hat{A}_3$  then transforms non-trivially under the gauge transformations associated with the 2-form potential  $A_2$ . The various gauge transformations cannot be completely disentangled and in fact the standard choice is instead to define  $\tilde{A}_3, A_2$  by

$$A_3 = \frac{1}{3!}\tilde{A}_{3abc}dx^a \wedge dx^b \wedge dx^c + \frac{1}{2!}A_{2ab}dx^a \wedge dx^b \wedge d\theta. \quad (1.32)$$

As a result, the 3-form  $\tilde{A}_3$  is *not* invariant under  $A_1 \rightarrow A_1 + d\Lambda_0$  but instead transforms as  $A_3 \rightarrow A_2 \wedge d\Lambda_0$ . In addition, the gauge symmetry of the eleven dimensional  $A_3$  implies that there are 9+1 gauge symmetries  $A_3 \rightarrow A_3 + d\Lambda_2$  and  $A_2 \rightarrow A_2 + d\Lambda_1$  where  $\Lambda_n$  are arbitrary  $n$ -forms. From here on, we drop the tilde ( $\sim$ ) on  $\tilde{A}_3$ . The decomposition of the fermionic fields is similar, but we will not go into this in detail.

### 1.4.3 On 9+1 dynamics: Here comes the dilaton

The dynamics for the 9+1 theory follows from that of eleven dimensions by inserting the relations between the 9+1 fields and the 10+1 fields into the action. The result is an action principle for the 9+1 theory which takes the form

$$\begin{aligned}
S_{9+1, \text{bosonic}} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \left[ \sqrt{-g} \left( e^{2\phi/3} R - \frac{1}{2} e^{2\phi} |F_2|^2 \right) \right. \\
& - \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left( e^{-2\phi/3} |F^2|_3 + e^{2\phi/3} |\tilde{F}_4|^2 \right) \\
& \left. - \frac{1}{4\kappa_{10}^2} \int A_2 \wedge F_4 \wedge F_4 \right]. \tag{1.33}
\end{aligned}$$

Here all quantities refer to the 9+1 dimensional fields and we have defined  $F_n = dA_{n-1}$  and  $\tilde{F}_4 = dA_3 - A_1 \wedge F_3$ . As opposed to  $F_4$  itself, the new field strength  $\tilde{F}_4$  is invariant under gauge transformations of the  $A_1$  potential. We have also defined  $\kappa_{10}^2 = \kappa_{11}^2/L$ .

An important feature of (1.33) is that the field  $\phi$  appears all over the place, with different factors of  $e^\phi$  appearing in different terms. The upshot of this is that the various gauge fields do not couple minimally to the metric  $g$ . Of course, we have the freedom to mix the metric with  $\phi$  by rescaling the metric by some power of  $e^\phi$ . This can be used to make any one of the gauge fields couple minimally to the new metric, or to remove the factors of  $e^\phi$  in front of the scalar curvature term and put the action in a form more like that of familiar Einstein-Hilbert gravity. However, because of the way that different factors of  $e^\phi$  appear in the different terms, this cannot be done for all fields at once. Thus, we may think of each different gauge field as coupling to a different metric.

A short calculation shows that the gauge fields  $F_2$  and  $F_4$  couple minimally to  $e^{2\phi/3}g$  while the gauge field  $F_3$  couples minimally to  $e^{-\phi/3}g$ . In doing this calculation, it is important to realize that terms like  $|F_2|^2$  contain implicit factors of the metric  $g$  (see 1.7) which has been used to contract the indices. On the other hand, it is for the ‘Einstein metric’  $e^{\phi/6}g$  that the gravitational part of the action takes the standard Einstein-Hilbert form (the integral of the scalar curvature density) without any extra factors of  $e^\phi$ .

The choice of a particular metric in the class  $e^{\alpha\phi}g$  is known as the choice of conformal frame. One can make a choice of frame that simplifies a given calculation, if one desires. It is interesting to note that in the conformal frame defined by (1.31) the field  $\phi$  has no explicit kinetic term so that its variation leads to a constraint. It turns out that this is just a combination of the usual constraints that one would expect in a gravitating theory, and that a term of the form  $\partial_a \partial^a \phi$  does appear in the equations of motion obtained by varying the metric in that frame.

The two most useful choices of conformal frame are the Einstein frame (defined by the Einstein metric  $g_E = e^{\phi/6}g$  discussed above) and the so-

called string frame. The action in the Einstein frame is a handy thing to have on hand, so we will write it down here. If we now let  $g_E$  denote the metric in the Einstein frame and let  $R_E$  be the associated curvature, the action is

$$\begin{aligned} S_{\text{IIA,bosonic}} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_E} \left( R_E - \frac{1}{2} \partial_a \phi \partial^a \phi \right) \\ & - \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g_E} \left( e^{3\phi/2} |F_2|^2 + e^{-\phi} |F^2|_3 + e^{\phi/2} |\tilde{F}_4|^2 \right) \\ & - \frac{1}{4\kappa_{10}^2} \int A_2 \wedge F_4 \wedge F_4. \end{aligned} \quad (1.34)$$

Note that, in Einstein frame, the gauge fields are all sources for the dilaton but the metric is not. Also, since the kinetic term for the dilaton now takes the standard form, we can see that the dilaton would be canonically normalized if we set  $2\kappa_{10}^2$  to one. Finally, since it is in this frame that the gravitational dynamics takes the familiar Einstein-Hilbert form, this is the frame in which the standard ADM formulas for energy and momentum may be applied and in which the entropy of black holes is given by  $A/4$  in Planck units.

The string frame is defined by taking the metric to be  $e^{2\phi/3}g$ , where  $g$  is the original metric that appeared in (1.31). Thus the string metric  $g_S$  and the Einstein metric  $g_E$  are related by  $ds_E = e^{-\phi/2}ds_S$  and the action in the string frame takes the form

$$\begin{aligned} S_{\text{IIA,bosonic}} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_S} e^{-2\phi} \left( R_S + 4\partial_a \phi \partial^a \phi - \frac{1}{2} |F_3|^2 \right) \\ & - \frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g_S} \left( |F_2|^2 + |\tilde{F}_4|^2 \right) \\ & - \frac{1}{4\kappa_{10}^2} \int A_2 \wedge F_4 \wedge F_4. \end{aligned} \quad (1.35)$$

After setting  $c = \hbar = 1$ , the parameter  $\kappa_{10}^2$  has units of  $(length)^8$ . It is useful to write  $2\kappa_{10}^2 = (2\pi)^7 g_s^2 l_s^8$  where  $l_s$  is the “string length” and  $g_s$  is the “string coupling.” For more on the separate role of  $g_s$  and  $l_s$ , see section 1.6.

Note that two of the gauge fields ( $F_2$  and  $F_4$ ) couple minimally to the string metric  $g_S$ . These two gauge fields are known as *Ramond-Ramond* (R-R) gauge fields while  $F_3$  is known as the *Neveu-Schwarz Neveu-Schwarz* (NS-NS) gauge field, or sometimes just as the Neveu-Schwarz (NS) gauge field for short<sup>16</sup>. For an explanation of how this terminology arose in string perturbation theory, see e.g. [5, 6, 7, 9]. The potential  $A_2$  for this field is

<sup>16</sup> In a confusing piece of terminology, it is sometimes also called the Kalb-Ramond gauge field.



commonly written  $B_2$  (and its field strength  $F_3$  is written  $H_3$ ) and when string theorists discuss “the B-field,” it is this potential to which they are referring.

What makes the string metric especially useful is that it turns out to be the metric to which fundamental strings (which we have not yet discussed) couple minimally, and thus in which one makes the most direct contact with string perturbation theory. This, however, is a discussion for another place and time.

In the above, we have discussed only the compactification of eleven dimensional supergravity on a circle. One can, of course, consider further compactifications to smaller dimensional manifolds. The story in that case is much the same except that the number of lower-dimensional fields generated increases rapidly. In particular, further compactification generates large numbers of massless scalars that couple non-minimally to the various gauge fields. These cousins of the dilaton are generally referred to as *moduli*.

All of these moduli have a tendency to diverge at the horizon of an extreme black hole, making the solution singular. One may think of the issue as follows: the moduli, like the dilaton, couple to the gauge fields so that the squared field strengths  $F^2$  act as sources. This can be seen from the action (1.34) in the Einstein frame. Non-singular extremal black hole solutions typically have an infinite throat, as in the four and five dimensional Einstein-Maxwell examples discussed earlier. This means that a smooth such solution would have an infinite volume of space near the horizon in which the gauge field strengths are approximately constant. Unless these gauge fields are tuned to have  $F^2 = 0$  or the various gauge fields are somehow played off against one another, this provides an infinite source for the moduli. As a result, smooth solutions are obtained only when the charges of the black hole are such that the potential for the moduli provided by the various  $F^2$  terms has a stationary point. In the spacetime solution, the moduli then approach this stationary point as one approaches the horizon. This phenomenon is known as the attractor mechanism for extreme black holes, see e.g. [46, 47]. As a result of this effect, some care is required to construct an extremal black hole solution with a smooth horizon and such solutions necessarily carry more than one charge. For a brane solution, the norm of each spacelike Killing field acts like a modulus whose sources must be properly tuned (as the norm would define a new dilaton-like scalar under further Kaluza-Klein reduction).

This is essentially the issue encountered at the end of section 1.3.3 in which it was found that three charges (in the case, three different types of M2-branes) were required to obtain a brane solution in which the norms of the

spacelike Killing fields did not vanish on the horizon. Recall that a Killing field with positive norm allows us to Kaluza-Klein reduce the spacetime to a solution of lower-dimensional supergravity. Because the three charge solution (1.26) has six Killing vector fields whose norms do not vanish on the horizon, it may be reduced all the way down to a solution of 4+1 gravity. In this context, it represents an extreme black *hole*. In fact, it reduces to just the standard 5+1 extremal black hole (1.20) of Einstein-Maxwell theory.

By the way, the theory discussed above is far from the only supergravity theory in ten dimensions. It is a particular kind called ‘type IIA,’ originally constructed in [43, 44, 45]. The ‘II’ refers to the fact that there are two independent gravitino fields and, as a result, two 10-dimensional spinors worth of supercharges. Each 10-dimensional spinor has 16 components, so this theory is maximally symmetric just like the 11-dimensional theory. In type IIA theory, these gravitinos have opposite chirality. This in turn allows type IIA theory to be defined even on non-orientable manifolds. There is also a type IIB theory which has two gravitinos (and is thus also maximally supersymmetric), but of the same chirality. Thus, type IIB theory can only be defined on manifolds with a global notion of chirality and, in particular, only on orientable spacetimes. We will discuss type IIB theory further in section 1.5.3 below. Two other supergravity theories with less supersymmetry (only 16 supercharges) are known as the type I and heterotic theories. Each of these types of supergravity in ten dimensions is associated with its own version of string theory. We will not discuss type I or heterotic supergravity here, but a discussion of these theories and how they are related to the type II theories can be found in [5, 6, 7, 8, 9].

### 1.5 Branes in 9+1 type II supergravity

We now wish to discuss the basic brane solutions of type II supergravity in 9+1 dimensions. Since any solution of type IIA theory is really a solution of eleven-dimensional supergravity in disguise, any brane solution of type IIA theory immediately defines a brane solution of eleven dimensional supergravity. Thus, we should be able to construct the basic brane solutions of type IIA theory by working with the basic brane solutions of section 1.3.2. For this reason we address the type IIA solutions first in section 1.5.1. Next follows a short aside on brane singularities in section 1.5.2. We then briefly discuss type IIB supergravity, and its relation through so-called T-duality with the type IIA theory, in section 1.5.3. Below, we will discuss only branes which become BPS in the extreme limit, although intrinsically non-BPS D-branes can also be of interest (see e.g. [48, 49]).

### 1.5.1 The type IIA branes

In our decomposition of the eleven-dimensional metric and gauge field into the various fields of ten dimensional supergravity, we proceeded by projecting the fields along directions both parallel and transverse to the Kaluza-Klein Killing field  $\lambda^\mu$ . In order to get brane solutions of type IIA theory that are charged under all of the type IIA gauge fields, a similar operation will need to be performed on the 11-dimensional branes. For any given brane in eleven dimensions, we will need to reduce both a basic brane solution in which the Killing field acts along the brane (i.e., is a symmetry of the brane), and one in which it acts transverse to the basic brane.

One may at first wonder what it means for the brane to be transverse to the Killing field since translations along a Killing field must leave the solution invariant, and therefore must preserve the brane. The answer to this puzzle is the smearing mentioned in section 1.3.3. One can take a basic brane solution, pick a direction transverse to the brane, smear the brane in that direction, and then reduce to 9+1 dimensions along the smearing direction.

Performing the required reductions amounts to no more than using the relations between the 9+1 fields and the 10+1 fields given in section 1.4 to write down the 9+1 solutions from the branes given in section 1.3.2. We leave the details of the calculations to the reader, but we provide a list here of the various 9+1 brane solutions. Below, we group together those branes charged under the Ramond-Ramond gauge fields and those charged under the NS-NS gauge fields. This grouping is natural from the point of view of the type IIA theory (and of string perturbation theory), though we will see that it is somewhat less natural from the eleven dimensional point of view.

Let us begin with the Ramond-Ramond branes. It turns out that the type IIA theory has  $p$ -brane solutions with Ramond-Ramond charge for every even  $p$ . What is very nice is that, in terms of the string metric, all of these solutions take much the same simple form. In order to treat all of the branes at once, it is useful to introduce a uniform notation for both electrically and magnetically charged branes. For each gauge field  $A_n$ , we can introduce (at least locally) a magnetic dual gauge field  $A_{9-n}$  through<sup>17</sup>  $dA_{8-n} = \star F_{n+1}$ . A brane which couples magnetically to  $A_n$  then couples electrically to  $A_{9-n}$  and vice versa. In type IIA theory, this notation should introduce no confusion as the standard gauge fields have  $n = 1, 2, 3$  while these new (dual) gauge fields have  $n = 5, 6, 7$ .

<sup>17</sup> Here we again ignore the Chern-Simons terms, which modify and complicate this simple uniform expression. As usual, this suffices for the solutions discussed below.

Introducing the usual set of  $p+1$  coordinates  $x_{\parallel}$  along the brane and  $9-p$  coordinates  $x_{\perp}$  transverse to the brane we have, for all even  $p$ ,

$$\begin{aligned} ds_{string}^2 &= H_p^{-1/2} dx_{\parallel}^2 + H_p^{1/2} dx_{\perp}^2 \\ A_{p+1} &= H_p^{-1} dx_{\parallel}^0 \wedge \dots \wedge dx_{\parallel}^p \\ e^{2\phi} &= H_p^{(3-p)/2}, \end{aligned} \quad (1.36)$$

where  $H_p$  is a function only of the  $x_{\perp}$  coordinates and satisfies

$$\partial_{\perp}^2 H_p = -(7-p)\Omega_{8-p} r_0^{7-p} \delta^{(9-p)}(x_{\perp}) \quad (1.37)$$

for the basic brane solution. Here  $\Omega_{8-p}$  is the volume of the unit  $(8-p)$ -sphere and  $r_0$  is a length scale parametrizing the strength of the source. These are the solutions known as extreme R-R  $p$ -branes or, in a slight abuse of language, as (extreme)  $Dp$ -branes, where the notation “D” comes from the way these objects are described in string perturbation theory (where they are associated with Dirichlet boundary conditions for strings, see section 1.6.2). As usual, we also obtain a solution by considering more general source terms on the RHS of eq. (1.37). For odd  $p$ , there are no gauge fields  $A_{p+1}$  in type IIA supergravity so (1.36) does not yield a solution to this theory for such cases<sup>18</sup>.

Let us briefly mention that the non-extreme R-R brane solutions (for  $p \leq 6$ ) take the rather simple form [50]

$$\begin{aligned} ds_{string}^2 &= H_p^{-1/2} \left( -f dt^2 + \sum_{i=1}^p (dx_{\parallel}^i)^2 \right) + H_p^{1/2} \left( \frac{dr^2}{f} + r^2 d\Omega_{8-p}^2 \right) \\ A_{p+1} &= [1 + \coth \beta (H_p^{-1} - 1)] dx_{\parallel}^0 \wedge \dots \wedge dx_{\parallel}^p \\ e^{2\phi} &= H_p^{(3-p)/2}, \end{aligned} \quad (1.38)$$

where

$$H_p = 1 + \frac{\sinh^2 \beta r_+^{7-p}}{r^{7-p}}, \quad f = 1 - \frac{r_+^{7-p}}{r^{7-p}} \quad (1.39)$$

and  $r_+, \beta$  specify the charge  $Q$ , mass  $M$  per unit  $p$ -volume  $V_p$ , temperature  $T$ , and entropy density  $S/V_p$  through

$$Q = \frac{(7-p)\Omega_{8-p} r_+^{7-p} \sinh \beta \cosh \beta}{2\kappa_{10}^2}$$

<sup>18</sup> One might also ask about the case  $p=8$ , since we have not discussed a 9-form gauge potential. It turns out that there is in fact a Ramond-Ramond 8-brane in type IIA theory and that its existence is tied to the Chern-Simons term in the type IIA action. In this work, we follow a policy of considering only the asymptotically flat brane solutions, which restricts us to branes of co-dimension 3 or higher; i.e., to  $p \leq 6$  in 10-dimensions.

$$\begin{aligned}
M/V_p &= \frac{(8-p)\Omega_{8-p}r_+^{7-p}}{2\kappa_{10}^2} \left(1 + \frac{7-p}{8-p} \sinh^2 \beta\right) \\
T &= \frac{7-p}{4\pi r_+ \cosh \beta} \\
S/V_p &= \frac{4\pi\Omega_{8-p}}{2\kappa^2} \cosh \beta r_+^{(8-p)}.
\end{aligned} \tag{1.40}$$

In particular, the extremal limit is  $\beta \rightarrow \infty$ ,  $r_+ \rightarrow 0$  with  $M/V_p$  fixed so that  $M/V_p \rightarrow Q$  and  $S \rightarrow 0$ . For  $p < 5$ , the temperature also vanishes in the extremal limit. However for  $p = 5$ ,  $T$  remains nonzero, and for  $p = 6$  it diverges. Here we allow only a single brane (analogous to having a single delta-function source in 1.37) as non-extremal branes attract each other and solutions with more than one such brane are not stationary. As for the M-branes, their global structure of (1.38) is like that of the Schwarzschild solution as opposed to that of non-extremal Reissner-Nordström.

Although all of these R-R branes take the same simple form (1.36), they proceed by quite different routes from the eleven dimensional branes. A short list follows: The D0-brane solution follows by reducing the smeared Aichelberg-Sexl metric along the smearing direction. The D2-brane follows by reducing the smeared M2-brane along the smearing direction. The D4-brane is the reduction of the unsmeared M5-brane in a direction along the brane. Finally, the D6-brane is the reduction of the unsmeared Kaluza-Klein monopole along the  $S^1$  fibers.

Next, there are the Neveu-Schwarz branes. Since the only Neveu-Schwarz gauge field is  $A_2$ , we expect to find two types of Neveu-Schwarz branes. The gauge field  $A_2$  should couple electrically to a 1-brane (a string) and it should couple magnetically to a 5-brane. The 1-brane follows by reducing the M2-brane in a direction along the brane. The resulting solution

$$\begin{aligned}
ds_{string}^2 &= H_F^{-1} dx_{\parallel}^2 + dx_{\perp}^2 \\
A_2 &= H_F^{-1} dx_{\parallel}^0 \wedge dx_{\parallel}^1 \\
e^{2\phi} &= H_F^{-1}
\end{aligned} \tag{1.41}$$

is known as the *fundamental string*. The reason for this is that this solution represents the classical limit of a long, straight version of the same string that appears in string perturbation theory.

The Neveu-Schwarz 5-brane (NS5-brane) is constructed by smearing the M5-brane in a transverse direction and then reducing along the smearing direction. The result is

$$ds_{string}^2 = dx_{\parallel}^2 + H_5 dx_{\perp}^2$$

$$\begin{aligned}
F_3 &= -\frac{1}{3!} \partial_{x_\perp^i} H_5 \epsilon_{ijkl} dx_\perp^j \wedge dx_\perp^k \wedge dx_\perp^l \\
e^{2\phi} &= H_5.
\end{aligned}
\tag{1.42}$$

An interesting property of the NS5-brane is that, in the string metric, the timelike Killing field has no horizon; its norm is constant across the spacetime. The would-be horizon at  $x_\perp = 0$  has receded to infinite proper distance in all directions, not just along a Killing slice as for the extreme Reissner-Nordström black hole. As a result, the coordinate patch above actually covers a manifold that, in the string frame, is geodesically complete<sup>19</sup>.

Finally, there are the purely gravitational ‘branes’ given by the 9+1 versions of the Aichelburg-Sexl metric and of the Kaluza-Klein monopole with all gauge fields set to zero and constant dilaton. These may be either written down directly by analogy with (1.23) and (1.24) or constructed by reducing the 10+1 solutions to 9+1 dimensions (and first smearing the Aichelberg-Sexl metric in some  $x_\perp$  direction).

This exhausts the possible ways to make extremal 9+1 branes by reducing (and perhaps smearing once) the basic eleven-dimensional branes. Below, we provide a few words on their global structure and singularities.

### 1.5.2 On brane singularities

We have constructed the D-brane spacetimes from what (in most cases) are smooth eleven-dimensional solutions (at least outside some horizon). However, the reduced solutions contain new singularities. Of the 9+1 branes, only the NS 5-brane does not have a naked singularity<sup>20</sup>. For the D4- and D6-branes and the fundamental string, this happens because the Killing field used in the reduction has fixed points so that  $\lambda^\mu \lambda_\mu$  and thus  $e^\phi$  vanishes (and  $\phi \rightarrow -\infty$ ).

For the D6-brane, the (6+1)-plane of fixed points at  $x_\perp = 0$  is manifest in (1.24) and results in a 6+1 dimensional timelike singularity in the 9+1 dimensional D6-brane solution. The conformal diagram for the D6-brane is therefore the one given in Fig. 11.5 below.

<sup>19</sup> However, it is not geodesically complete either in the Einstein frame or as viewed from the eleven-dimensional perspective. In each of these cases, there is a null singularity at the horizon.

<sup>20</sup> This statement refers to the metric in the string frame. In the Einstein frame there is a naked singularity on the horizon. Its story is much like that of the D2-brane discussed below.

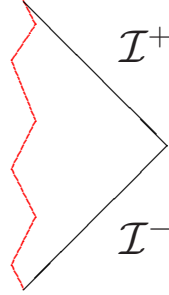


Figure 11.5 Conformal diagram for the extreme D6-brane.

Turning to the D4 brane, the fixed points of the Kaluza-Klein Killing field in the compactified M5-brane solution are less obvious from (1.22) but were briefly discussed in section 1.3.2. There we saw that the fixed point set consisted of certain horizon generators and so defined a null plane. As a result, the singularity of the D4-brane is null and lies on the would-be horizon. Because  $\phi \rightarrow -\infty$ , in terms of the 9+1 metric this turns out to be a (null) curvature singularity. The story of the fundamental string is much the same. See Fig. 11.6 for both.

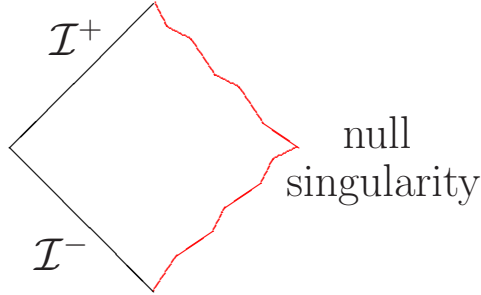


Figure 11.6 Conformal diagram for the extreme D4-brane or fundamental string.

Let us now consider the D2-brane solution, which is the reduction of a smeared M2-brane. Although smearing the M2-brane in a transverse direction makes the horizon of the eleven dimensional solution singular, one may take the perspective that the smeared solution represents the approximate solution for an array of M2-branes for which the length scale associated with the charge of each brane is much larger than the spacing between the branes. In this case, one interprets the D2-brane horizon as being non-singular<sup>21</sup> from the eleven dimensional point of view. However, from the 9+1 perspective, we clearly have  $\phi \rightarrow +\infty$  on the horizon. The metric (1.36) with  $p = 2$  also has a null curvature singularity on the horizon, though all curvature scalars are finite. The simplest way to detect the singularity from the metric

<sup>21</sup> Actually, as consisting of many separate non-singular horizons.

(1.36) is to note that the spheres around the brane shrink to zero size at the horizon so that, if the solution were smooth, the horizon could have only a single null generator, which is impossible. Thus the conformal diagram of the 9+1 D2-brane solution is again given by Fig. 11.6.

With an eleven dimensional perspective in mind, the singularities of the D2-, D4-, and D6-brane solutions might not be considered especially troubling. Nevertheless, from the 9+1 perspective the singularities are quite real and represent places where the 9+1 equations of motion break down. Let us recall that the Aichelberg-Sexl metric (the lift of the D0-brane solution to eleven dimensions) *is* singular and should be thought of as describing the approximate field produced by some ‘source.’ For the Aichelberg-Sexl metric, one may think of this source as being a short wavelength graviton, with the solution (1.23) itself representing just the Coulomb part of the field. Similarly, looking at the way that the D-brane singularities interact with the equations of motion through (1.37), it is natural to think of the singularities as representing bits of matter, like a braney form of extremal dust, which are coupled to the supergravity. By the way, this dual perspective of thinking of branes either as solitonic objects intrinsic to some basic version of the theory (like supergravity or string theory) or as external objects or sources coupled to such a theory is pervasive in string/M-theory.

### 1.5.3 The type IIB theory and *S*- and *T*-dualities

The other type of maximally supersymmetric gravity theory in 9+1 dimensions is called type IIB theory (originally constructed in [51, 52, 53]). It is not given by the dimensional reduction of a 10+1 theory, though it has many of the same properties as the type IIA theory. For example, both theories consist of a metric, a dilaton, and a  $B_2$ -field (which together form the so-called Neveu-Schwarz sector of the theory) together with various Ramond-Ramond gauge fields. The theories are identical in the Neveu-Schwarz sectors, so that the Aichelberg-Sexl, fundamental string, NS5-brane, and Kaluza-Klein monopole solutions are the same in both cases. The main difference is that while IIA supergravity has (electric and magnetic) Ramond-Ramond gauge fields  $A_p$  of every odd rank, the type IIB theory has Ramond-Ramond gauge fields  $A_p$  of every even rank. As in IIA, the Ramond-Ramond fields are minimally-coupled to the string-frame metric. As a result, the  $Dp$ -brane solutions in IIB are again given by (1.36), though now  $p$  is odd; i.e., the IIB theory has D1, D3, D5, D7, and D9-branes<sup>22</sup>.

<sup>22</sup> If one is interested in Euclidean solutions then it also makes sense to consider D(-1)-branes (D-instantons) in the IIB theory.



The D3-brane has particularly noteworthy features. Since the Ramond-Ramond fields are minimally coupled in the string frame, the kinetic term of  $A_4$  is of the form  $\sqrt{-g_S}|F_5|^2$ . Note that this term is conformally invariant in 10-dimensions (just as the usual Maxwell kinetic term  $\sqrt{-g}|F_2|^2$  is conformally invariant in 4-dimensions). As a result,  $A_4$  remains minimally coupled in any conformal frame, and in particular in the Einstein frame where the kinetic terms of the graviton and the dilaton have been diagonalized. Thus, in contrast to all other IIB gauge fields, the 5-form  $F_5$  does not act as a source for the dilaton! Indeed, consulting (1.36) one finds that  $\phi = \text{constant}$  for  $p = 3$ . As a result the D3-brane has a smooth horizon. In fact, the solution very similar to that of the M5-brane, having the same singularity-free conformal diagram (Fig. 11.4). The near-horizon geometry is  $\text{AdS}_5 \times S^5$ .

Clearly the D3-brane is the marginal case that separates  $Dp$ -branes with  $p < 3$  from those with  $p > 3$ . As one might expect, the D1 (and D(-1)) solution resembles that of the IIA D0,D2 solutions (with a null singularity where  $\phi \rightarrow +\infty$ ) while the D5 resembles the D4 (with a null singularity where the Ricci scalar diverges). Branes with large  $p = 7, 9$  are special cases which we ignore here as they are not asymptotically flat.

To fully understand the relations between the various brane solutions it is important to understand certain symmetries known as S- and T-duality. We begin with S-duality, which is a symmetry of IIB supergravity alone. To study this symmetry, it is useful to define a complex scalar  $\tau = A_0 + ie^{-\phi}$ , a  $2 \times 2$  matrix  $\mathcal{M}_{ij} = \frac{1}{\text{Im } \tau} \begin{bmatrix} |\tau|^2 & -\text{Re } \tau \\ -\text{Re } \tau & 1 \end{bmatrix}$ , and a vector of 2-form potentials  $A_2^i = \begin{bmatrix} B_2 \\ A_2 \end{bmatrix}$ , where we have called the Neveu-Schwarz gauge field  $B_2$  (with field strength  $H_3$ ) to distinguish it from the Ramond-Ramond gauge field  $A_2$  (with field strength  $F_3$ ). In terms of these fields, the Einstein frame action<sup>23</sup> takes the form [7]

$$S_{\text{IIB,bosonic}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_E} \left( R_E - \frac{1}{12} F_{abc}^i \mathcal{M}_{ij} F^{jabc} + \frac{1}{4} (\partial^a \mathcal{M}_{ij} \partial_a \mathcal{M}^{-1ij}) \right) - \frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-g_S} |\tilde{F}_5|^2 - \frac{1}{4\kappa_{10}^2} \int A_4 \wedge H_3 \wedge F_3, \quad (1.43)$$

where  $F_3^i = \frac{1}{3!} F_{abc}^i dx^a \wedge dx^b \wedge dx^c = dA_2^i$  and  $\tilde{F}_5 = F_5 + \frac{1}{2} \epsilon_{ij} A_2^i \wedge F_3^j$  with  $\epsilon_{ij}$  antisymmetric and  $\epsilon_{12} = 1$ .

The action (1.43) makes manifest an invariance under the  $\text{SL}(2, \mathbb{R})$  sym-

<sup>23</sup> The equations of motion that follow from this action must be supplemented by the constraint that the 5-form  $F_5 = dA_4$  is self-dual; i.e., that  $\star F_5 = F_5$ .

metry [51]

$$\begin{aligned} \tau &\rightarrow \frac{a\tau + b}{c\tau + d}, & F_3^i &\rightarrow \Lambda_j^i F_3^j, \\ \tilde{F}_5 &\rightarrow \tilde{F}_5, & g_E &\rightarrow g_E, \end{aligned} \quad (1.44)$$

for  $\Lambda_j^i = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$  with  $\det \Lambda = 1$ . The transformation with  $\Lambda_j^i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  is of particular interest and is known as S-duality. For solutions with  $A_0 = 0$ , it acts as  $\phi \rightarrow -\phi$  and interchanges  $F_3$  and  $H_3$ . As a result, it maps D1-branes to fundamental strings and maps D5-branes to NS5-branes (and vice versa). On the other hand, it leaves the D3-brane and all purely gravitational branes invariant.

Finally, we turn to T-duality, which is a symmetry that relates the full type IIA and IIB string theories. At the level of supergravity, this is a symmetry that maps solutions of type IIA theory with a Killing field into solutions of type IIB theory with a Killing field. It appears that T-duality is an exact symmetry of the underlying string/M-theory even with no Killing field, but that it then maps a nearly classical spacetime into a complicated highly quantum mechanical state.

It is useful to first write down the explicit action of T-duality on the metric and Neveu-Schwarz fields. Let us introduce a coordinate  $z$  such that translations in  $z$  are a Killing symmetry. Let  $x^\alpha$  be any other collection of coordinates which makes  $(z, x^\alpha)$  a coordinate patch. Again writing the anti-symmetric Neveu-Schwarz field as  $B_2$  instead of  $A_2$ , if the original solution is  $(g, B)$  then the transformed solution  $(\tilde{g}, \tilde{B})$  is given [54] by

$$\begin{aligned} \tilde{g}_{zz} &= 1/g_{zz}, & \tilde{g}_{z\alpha} &= B_{z\alpha}/g_{zz}, \\ \tilde{g}_{\alpha\beta} &= g_{\alpha\beta} - (g_{z\alpha}g_{z\beta} - B_{z\alpha}B_{z\beta})/g_{zz}, & \tilde{B}_{z\alpha} &= g_{z\alpha}/g_{zz}, \\ \tilde{B}_{\alpha\beta} &= B_{\alpha\beta} - (g_{z\alpha}B_{\beta z} - g_{z\beta}B_{\alpha z})/g_{zz}, & \tilde{\phi} &= \phi + \log g_{zz}. \end{aligned} \quad (1.45)$$

Note that T-duality essentially interchanges the  $g_{z\alpha}$  part of the metric with the  $B_{z\alpha}$  part of the gauge field<sup>24</sup>. Now, in the asymptotically flat context, the  $g_{z0}$  component of the metric is associated with momentum in the  $z$ -direction while the  $B_{z0}$  component of the gauge field is associated with electrically charged strings that extend in the  $z$ -direction. Thus one finds [55] that T-duality interchanges momentum and charge, and in particular maps the (one-smeared) Aichelburg-Sexl solutions to fundamental string solutions (which carry the electric charge to which  $B$  couples).

<sup>24</sup> In fact, in the case where the  $z$ -direction is compactified into an  $S^1$ , (1.45) can be described geometrically by noting that the Kaluza-Klein reduction to 8+1 dimensions has two 1-form gauge fields, one from the 10-dimensional metric and one from the 10-dimensional  $B_2$ . In the Neveu-Schwarz sector, T-duality simply interchanges the two circle-bundles associated with these gauge fields.

Strictly speaking, the T-duality of string theory requires a Killing field with compact ( $S^1$ ) orbits, through (1.45) maps solutions to solutions even if the orbits are non-compact. The original spacetime should be asymptotically Kaluza-Klein and, if the original  $z$  coordinate is identified such that the length of the  $S^1$  at infinity is  $L$ , then the  $z$  coordinate of the transformed spacetime should be identified such that the length of the  $S^1$  at infinity is  $\frac{4\pi^2 l_s^2}{L}$ . The point is that, if the orbits of the Killing field are compact, then quantum mechanics implies that the momentum component around the compact direction is quantized. Proper normalization then guarantees that T-duality takes a solution with one quantum of momentum to a solution containing a single fundamental string.

The effect of T-duality on the Ramond-Ramond fields is as follows:

$$\begin{aligned}\tilde{F}_{n,\alpha_1\ldots\alpha_n} &= (const) F_{n+1,z\alpha_1\ldots\alpha_n}, \\ \tilde{F}_{n,z\alpha_1\ldots\alpha_{n-1}} &= (const) F_{n-1,\alpha_1\ldots\alpha_{n-1}}.\end{aligned}\tag{1.46}$$

Thus, if one takes a  $Dp$ -brane and T-dualizes in some direction along the brane, one obtains a  $D(p-1)$ -brane solution which is smeared along the T-duality direction. In string theory, D-brane charge is quantized. The normalization constants in (1.46) are chosen so that the transformed solution has one unit of  $D(p-1)$ -brane charge when the original solution has one unit of  $Dp$ -brane charge. Similarly, if one smears a unit charge  $Dp$ -brane solution in a transverse direction keeping the total charge equal to one quantum, the T-dual solution is a unit charge  $D(p+1)$ -brane. See [5, 6, 7, 8, 9] for a discussion of D-brane tensions, charge quantization, etc. In any given direction, one may check that the T-duality transformation squares to the identity.

## 1.6 Some remarks on string perturbation theory

The preceding sections we have discussed the supergravity aspects of various string-theoretic branes, including D-branes. However, the real power of D-branes, and thus their importance, stems from the fact that there is a renormalizable (in fact, order by order finite) quantum perturbation theory to complement the classical supergravity description. This perturbation theory describes both the internal dynamics of D-branes and their interactions with the supergravity fields. In particular, it is the key to the famous counting of states of BPS and near-BPS black holes. Thus, although we will not discuss the details, it is worthwhile to say a few words here about this perturbation theory. We hope this gives a useful complement to standard presentations which concentrate more on the perturbation theory details.

### 1.6.1 Background field expansions and perturbative string theory

A useful framework from which to view string perturbation theory is that of the background field expansion (see e.g. [56]). Let us first review this idea in the context of standard quantum field theory. For definiteness, the reader may choose to focus on a familiar low dimensional interacting scalar field theory or even quantum mechanics. We will use  $\phi$  to denote the scalar field or, more generally, as a schematic notation for the collection of all relevant fields.

Let us begin by supposing that there is some complete quantum theory of this field, consisting of a set of field operators  $\hat{\phi}(x)$  and an associated set of composite operators acting on a Hilbert space. Exact calculations for interacting quantum field theories are seldom possible, and one must resort to various approximation schemes and expansions in small parameters in order to obtain results. For situations where the field is nearly in its vacuum state, standard perturbation theory (see e.g. [57]) can be a useful technique.

However, this is not the only case of interest. For example, it may be that a laboratory device (or a star, black hole, or astrophysical event) produces a large, essentially classical, disturbance in the field  $\phi$  and that one wishes to study small quantum effects in the resulting behavior. It is in such a regime that background field methods are useful. One first considers the solution  $\phi_0$  to the classical field equations that would describe the situation if  $\hbar$  were set to zero. One then rewrites the theory in terms of the field  $\widehat{\delta_0\phi}(x) = \hat{\phi}(x) - \phi_0(x)$ . Assuming that there is in fact a set of semi-classical states in which the expectation value of  $\hat{\phi}(x)$  is close to  $\phi_0(x)$  and in which the fluctuations are ‘small,’ it makes sense to attempt a perturbative treatment in terms of the field  $\widehat{\delta_0\phi}$ .

This is the basic idea behind the background field expansion. However, there is an additional subtlety. Although one expects any differences to vanish as  $\hbar \rightarrow 0$ , there need not be any state in which the expectation value of  $\hat{\phi}(x)$  is exactly  $\phi_0(x)$ . In a perturbative framework, one assumes that the difference between the actual expectation value and the classical solution  $\phi_0$  can be expanded in powers of  $\hbar$  and simply solves for it at each order of perturbation theory. It is useful to take the expectation value calculated at order  $n$ , which we write as  $\overline{\phi}_n$ , to be an effective ‘classical field’ and to work at order  $n$  with the perturbation  $\widehat{\delta_n\phi} = \phi(x) - \overline{\phi}_n$ .

Within the range of validity of this perturbation theory, one can (see [56]) expand about a general classical solution  $\phi_0$  and obtain, at order  $n$  in perturbation theory, an ‘effective action’ for the ‘effective classical background field’  $\overline{\phi}$ . The variations of the effective action with respect to the effective

field yield the classical equations of motion for  $\phi$  corrected by terms of up to order  $n$  in  $\hbar$  such that the solutions of these equations yield the expectation value of  $\hat{\phi}(x)$  (to order  $n$ ) in a semi-classical state.

One could also attempt to follow the same general framework but to expand around some arbitrary field  $\phi_0$  which is not a solution to the classical equations of motion. In this case, the field  $\widehat{\delta_0\phi}(x)$  is not small and the perturbation theory will not contain anything like a stable vacuum. Because the variation of the action does not vanish at the chosen background, the action contains a term linear in  $\widehat{\delta_0\phi}(x)$  which acts as a source. This typically leads to various infrared divergences in the perturbation theory since, when integrated over all time, this source will produce an infinite number of particles. Thus if, for some reason, someone had handed us not the full classical dynamics of the field but only the equations of the perturbation theory around an arbitrary background, the classical solutions of the theory would still be recognizable.

String perturbation theory is in fact a version of background field theory in which the ‘strings’ correspond to excitations of the field  $\widehat{\delta_0\phi}(x)$ . However, the logical order of the background field framework is reversed or, perhaps more accurately, turned inside out. Instead of starting with a classical theory, quantizing, and performing the background field expansion, one instead postulates the perturbative expansion<sup>25</sup> about any background field and then reconstructs the ‘classical’ dynamics of the background field in the manner discussed above from the condition that the perturbation theory is well-defined.

This seemingly odd logical structure makes more sense when one recalls that string theory is not, at present, a complete theory based on any particular set of fundamental principles or axioms. Rather, it is really an accidentally discovered set of self-consistent mathematical phenomena related to quantum gravity, the unification of forces, and so on. The way that string perturbation theory arose historically was through interest in QCD and possible ‘strings’ of gauge field flux that would connect quarks in hadrons. While studying such strings, it was discovered that they defined a perturbation theory which was finite order by order and which contained a spin two particle which could be interpreted as a graviton. Since finding a perturbative treatment of quantum gravity, or even constructing a new theory of gravity which could be treated perturbatively, had been a question of interest for some time, string theory presented a solution to this technical problem: Simply take this accidentally discovered perturbation theory and

<sup>25</sup> More accurately, the S-matrix corresponding to such an expansion.

use it to construct an associated theory of quantum (and classical) gravity. In the case of string theory, the postulated perturbation theory was used to construct not only the classical dynamics of the various fields, but also to deduce the classical field content itself. The rest, as they say, is history.

Our story of supergravity discussed in the previous sections is relevant here because the dynamics of string theory reduces in a certain limit to classical supergravity. A few fine points are worth mentioning briefly. The first is that, when viewed as a background field theory of the sort just discussed, classical string theory actually contains not only the fields of classical supergravity, but an infinite tower of massive fields as well. The masses of these *classical* fields are, however, on the order of  $l_s^{-1}$  (and therefore considered to be large). Thus, one expects there to be a large regime in which these fields are not independently excited. Instead, the heavy (massive) fields are ‘locked’ to the values of the massless fields. At the extreme end of this regime, the massive fields are completely irrelevant. However, as one pushes toward the boundaries of this regime, the massive fields may still have some effect on the dynamics. If one solves the classical equations of motion for the heavy fields, one finds that they are disturbed slightly by the massless fields and then in turn provide small sources for the massless fields. This analysis, known in the lingo of path integrals as ‘integrating out’ the massive fields, leads to additional effective interactions between the massless fields. Such interactions are non-local on a scale set by the masses of the heavy fields; i.e., on the scale of the string length. When expanded in a power series, they lead to a series of higher derivative terms in the action suppressed by powers of the string scale. These are the so-called  $\alpha'$ -corrections, where  $\alpha' \propto l_s^2$ .

In this way the string scale explicitly appears in the dynamics of classical string theory. Now, it is true that in the ‘real world’ the string length is likely to be within a few orders of magnitude of the Planck scale. In principle, however, the two scales are completely independent and should not be confused. The string scale controls the corrections to classical supergravity caused by the tower of massive fields and the (9+1) Planck scale is the true quantum scale. Their ratio defines the string coupling  $g_s$ . The regime in which string perturbation theory is useful is  $g_s \ll 1$ , in which the string length is much greater than the 9+1 Planck length.

### ***1.6.2 Strings and D-branes***

In order to describe how D-branes fit into this picture, we should say just a few more words about the relation of strings to supergravity. As mentioned above, strings provide rules for constructing the perturbation theory about

a given 9+1 supergravity background. Roughly speaking, one replaces the Feynman diagrams (related to particles) of familiar perturbation theory with a new sort of diagram related to strings. For details, the reader should consult [4, 5, 6, 7, 8, 9]. For most of our purposes below, it will suffice to think about the strings as classical objects.

One can conceive of two basic types of strings. The first are the so-called closed strings, which at any moment of time have the topology  $S^1$  and resemble a classical rubber band. It turns out that the closed strings define a consistent perturbation theory in and of themselves, and that it is this case that leads to the type II supergravities on which we have focused. Another version of the closed string leads to heterotic supergravity, which has half as much supersymmetry as the type II theories.

One might also consider so-called open strings which, at any instant of time, have the topology of an interval. In order for the dynamics of such strings to be well-defined, one must specify boundary conditions at the ends. A natural choice is to impose Neumann boundary conditions to describe free ends. Such strings are quite similar to classical rubber bands that have been cut open. It turns out that this type of string does not yield a consistent perturbation theory by itself, as two open strings can join together to produce a closed string. When open and closed strings are taken together, a consistent perturbation theory does result. This theory is associated with type I supergravity, having half as much supersymmetry as the type II theories.

The other type of boundary condition that one can impose at the end of a string is the Dirichlet boundary condition, requiring the end of the string to remain fixed at some point in space. One can also consider a mixture of Dirichlet and Neumann boundary conditions, insisting that the end of the string remain attached to some submanifold of spacetime, but otherwise leaving it free to roam around the surface. Surfaces associated with such Dirichlet boundary conditions are known as Dirichlet submanifolds, or D-branes. Again, for a consistent perturbation theory, one must consider closed strings in addition to these open strings. Since we have singled out this submanifold as a special place in the spacetime, this perturbation theory should not describe an expansion about empty space. However, there remains the possibility that it can describe an expansion about a background in which certain sub-manifolds are picked out as special; i.e., near a background which includes certain brane-like features. Recall that, as a background field expansion, this perturbation theory should tell us about all of the dynamics of the background, including any dynamics of the branes.

To make a long story short, it turns out that the Dirichlet submanifolds are sources of the Ramond-Ramond gauge fields and of the gravitational



field. That is, they carry both stress-energy and Ramond-Ramond charge. Thus, one might expect that they have something to do with the branes discussed earlier that carry Ramond-Ramond charge. In fact, this D-brane perturbation theory is supposed to give the expansion about a background that includes such a charged gravitating R-R brane in the regime of asymptotically small string coupling  $g_s$ , which controls the strength of all interactions. The perturbation theory describes both the dynamics of the bulk fields (roughly speaking, through the closed strings) and of the brane itself (roughly speaking, through the open strings). The two parts are coupled and interact.

Let us briefly comment on how this picture meshes with the supergravity point of view. To do so, we must consider the strength of the above interaction. This is determined by the source the brane provides for the various supergravity fields. But a simple brane has only one charge, which is necessarily equal to its mass per unit volume if the brane is BPS. As a result, the strength of the source is governed by  $GT$  where  $16\pi G = 2\kappa_{10}^2 = (2\pi)^7 g_s^2 l_s^8$  and  $T$  is the brane tension.

Now, in string theory, as in any theory with both electric and magnetic charges, the charge (and thus  $T$  for BPS branes) is quantized in integer multiples of some fundamental unit. It turns out that the charge of any R-R brane (with  $n$  units of charge) is proportional to  $n/g_s$ . Thus,  $GT \sim ng_s$  goes to zero at weak coupling. As a result, the supergravity fields go over to flat empty space in this limit and the field generated by a fixed number  $n$  of such branes is indeed perturbative at small  $g_s$ . On the other hand, since the mass per unit volume of the D-brane is diverging, any internal dynamics associated with motion of the D-brane is frozen out in this limit. Thus the picture from supergravity agrees with the string perturbation theory described above: up to perturbations, it consists of flat empty space with a preferred submanifold in spacetime occupied by a largely non-dynamical brane. This suggests that the D-branes of perturbation theory should be identified with the Ramond-Ramond branes of supergravity. Additional evidence for this picture comes from the great success of D-brane perturbation theory in reproducing the entropy of black holes [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68], Hawking radiation [69, 70, 71], and the so-called grey-body factors [72] associated with the Ramond-Ramond branes.

In contrast, the Neveu-Schwarz branes do not admit simple descriptions as backgrounds for string perturbation theory. For the fundamental string, this is because (with  $n$  units of charge) the tension is proportional to  $n$  and does not depend on the string coupling. They therefore remain fully dynamical at small  $g_s$  (while the spacetime solution again becomes just Minkowski space).



For the NS5-brane (with  $n$  units of charge) the tension is proportional to  $n/g_s^2$ . Thus,  $GT \sim n$  and the supergravity fields remain unchanged as we take  $g_s \rightarrow 0$ ; the fields cannot be described as small perturbations about Minkowski space.

### 1.6.3 A few words on black hole entropy

This is not the place for an in-depth discussion of just how D-brane perturbation theory can be used to reproduce the properties of supergravity solutions. Such treatments can be found in [73] and in [5, 6, 7, 9]. They involve the fact that the open strings associated with D-branes describe, in the low energy limit, a certain non-abelian Yang-Mills theory. The low energy limit of that theory can then be analyzed and used to study the low energy limit of the brane dynamics. Since BPS branes have the minimal possible energy for their charge, this means that BPS and nearly BPS branes can be addressed by such techniques.

We will, however, close by giving some parts of the entropy calculation for a particular case. As has already been mentioned, the solution (1.26) with three mutually orthogonal sets of M2-branes is the simplest BPS black brane solution with non-zero entropy. Let us compactify a circle along one of the M2-branes (say, the one associated with the  $z_{\parallel}$  coordinates) and Kaluza-Klein reduce to 9+1 type IIA supergravity. Then, as we have seen, the  $z$ -type M2-branes (which are wrapped around this circle) become fundamental strings in the IIA description while the  $x$ - and  $y$ -type M2-branes (which do not wrap around the compact circle) become D2-branes. It turns out that a simple description of the microscopic perturbative states can be obtained by T-dualizing this solution to the IIB theory along the direction in which the fundamental strings point. This turns the fundamental strings into momentum and the two sets of D2-branes into D3-branes.

Let us now T-dualize twice more in, say, the two  $y_{\parallel}$  directions. This again yields a solution of IIB theory. The momentum remains momentum in the same direction, but one of the sets of D3-branes has become a set of D1-branes and the other has become a set of D5-branes. The D1-branes (D-strings) are stretched in the same direction that the momentum is flowing, and this all happens in one of the directions along the D5-branes. These T-dualities do not change the integer charges  $Q_x, Q_y, Q_z$  associated with the various types of branes:  $Q_x$  is now the number of D5-branes,  $Q_y$  the number of D-strings, and  $Q_z$  the number of momentum quanta. One can check that these T-dualities do not change the Bekenstein-Hawking entropy

and, as supposed symmetries of the underlying string theory, they cannot change the number of microstates.

The case of a single D5-brane is particularly simple to discuss. It turns out that the low energy dynamics reduces to what is effectively just a collection of D-strings<sup>26</sup> which are stuck to the D5-brane but free to oscillate within it. The momentum in the solution is just the momentum carried by these oscillations, and the energy of the solution is a linear sum of contributions from the D5-brane rest energy, the D-string rest energy, and the momentum. For a supersymmetric solution, all of the oscillations must move in the same direction along the D-string and so are described by, say, right-moving fields on a 1+1 dimensional spacetime. Oscillations of D-strings propagate at the speed of light and so the associated energy-momentum vector is null.

Thus, for each string, one has 4 massless 1 + 1 rightmoving scalar fields corresponding to the four internal directions of the fivebrane. Supersymmetry implies that there are also four massless 1 + 1 rightmoving fermionic fields for each D-string. A fermion acts roughly like half of a boson, so we may think of this as  $6Q_y$  massless right-moving scalars on  $S^1 \times R$  (the world-volume of a one-brane). A standard formula tells us that, given  $n$  massless rightmoving scalars with  $Q_z$  units of momentum, the entropy at large  $Q_z$  is  $S = 2\pi\sqrt{Q_z n}/6$ . Thus we have  $S = 2\pi\sqrt{Q_y Q_z}$ , in agreement with the Bekenstein-Hawking entropy  $S = A/4G_{11}$  (see eq. (1.28) with  $Q_x = 1$ ) for the associated black hole.

This gives an idea of the way in which D-brane perturbation theory provides a microscopic accounting of the entropy of this BPS black hole. The other BPS and near-BPS cases are similar in many respects. It is quite satisfying to arrive at exactly the Bekenstein-Hawking entropy formula without having to adjust any free parameters. However, one is certainly struck by the qualitative differences between the regime in which we are used to thinking about black holes and the regime in which the string calculation is performed. We usually consider black holes with large smooth horizons. In contrast, the perturbative calculation is done in the asymptotic regime of small  $g_s$ , where spacetime is flat and the horizon has degenerated to zero size. The belief is that supersymmetry guarantees the entropy of the quantum system to be independent of  $g_s$ , as it does for other non-gravitational systems<sup>27</sup>. However, there is much room for speculation and investigation in

<sup>26</sup> Or, even better, to a single D-string wrapped  $Q_y$  times around the direction in which the momentum flows. See, e.g., [73].

<sup>27</sup> As supporting evidence, recall that the Bekenstein-Hawking entropy of our BPS black hole does not depend on  $g_s$  when written in terms of the integer charges (1.28).

trying to match these pictures more closely and in understanding just what form these states take in the black hole regime of finite  $g_s$ .

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